

AMENABILITY PROPERTIES FOR THE CENTRES OF CERTAIN DISCRETE GROUP ALGEBRAS

M. ALAGHMANDAN, Y. CHOI, AND E. SAMEI

27th October 2011

Abstract

Let G be a discrete group with finite conjugacy classes, and let $Z\ell^1(G)$ denote the centre of its group algebra. In this article, we study various properties of $Z\ell^1(G)$, such as amenability and its spectrum, for the case where G is a restricted direct product of finite groups $\{G_i\}_{i \in I}$. Among other things, we show that $Z\ell^1(G)$ is amenable if and only if G_i is abelian for all but finitely many i , and characterize maximal ideals of $Z\ell^1(G)$ with bounded approximate identities. We also study when an algebra character of $Z\ell^1(G)$ belongs to c_0 or ℓ^p and provide a variety of examples. Lastly, we calculate the exact amenability constant of $Z\ell^1(G)$ for certain finite metabelian groups G .

Keywords: Center of group algebras, restricted direct product of finite groups, characters, amenability, amenability constant, maximal ideals.

MSC 2010: 43A20 (primary); 20C15 (secondary).

CONTENTS

1	Introduction	2
2	Notation and other preliminaries	3
2.1	Important conventions	3
2.2	General properties of $Z\ell^1$	4
2.3	The restricted direct product of finite groups	6
3	Characterizing ZL-amenability of RDPF groups	8
4	Bounded approximate identities in maximal ideals of $Z\ell^1(G)$	9
5	c_0 and ℓ^p estimates for elements of $\text{sp}(Z\ell^1(G))$	15
5.1	Preliminary discussions	15
5.2	Characterizing when $\omega \in c_0$	16
5.3	Examples where $\omega \in \ell^p$ for various p	19

6	ZL-amenability constants for groups with two character degrees	22
7	ZL-amenability constants of particular groups	25
7.1	Dihedral groups	25
7.2	Frobenius groups with abelian complement and kernel	26
7.3	Extra-special p -groups.	30
A	Appendix: a proof that 1-minimal groups are nilpotent	31
B	Properties of Frobenius groups	32

1 INTRODUCTION

The L^1 -convolution algebra of a locally compact group G is amenable if and only if the group G is amenable. In contrast, there is as yet no ‘intrinsic’ characterization of those groups G for which $ZL^1(G)$, the centre of $L^1(G)$, is amenable. It will be convenient to call such groups **ZL-amenable**.

The article [1] studied the problem of which compact groups are ZL-amenable. One can consider the corresponding problem for (discrete) FC groups, that is, the groups in which each conjugacy class is finite. We note that the problem is solved for *finitely generated* FC groups: for, by a result of B. H. Neumann [11, Theorem 5.1], each such group has a finite derived subgroup; and when a discrete group has a finite derived subgroup, then it is ZL-amenable by a result of Stegmeir [14, Theorem 1].

The present paper is concerned with a particular class of infinitely generated FC groups, namely the restricted direct products of finite groups (**RDPF groups**, for short) whose formal definition is given below in Definition 2.6. The C^* and von Neumann algebras of discrete RDPF groups were studied in some old work of Mautner [8], but to our knowledge the centres of their ℓ^1 -group algebras have not been studied explicitly.

When G is such a group, we are able to study $Z\ell^1(G)$ in some detail. We show in Theorem 3.3 that an RDPF group is ZL-amenable if and only if its derived subgroup is finite. Then, in Theorem 4.2, we obtain a characterization of those RDPF groups G for which every maximal ideal in $Z\ell^1(G)$ has a bounded approximate identity. (For a commutative, unital Banach algebra, the condition that each maximal ideal has a bounded approximate identity can be thought of as a weaker version of amenability; it has been recently rediscovered and studied under the name of **character amenability**; see [6] and [9] for some directions in which this notion has been pursued.)

Using our techniques, in Section 5, we revisit the main counter-example from Stegmeir’s paper (*ibid.*), and are able to give simpler proofs of some of his results. In doing so, we are led to consider when given characters on $Z\ell^1(G)$ belong to c_0 , when viewed as functions on G in the natural way. We obtain a necessary and sufficient condition for this to occur, and also study the related question of when such characters lie in ℓ^p for various exponents $p \in [1, \infty)$.

Motivated in part by these calculations, in the last two sections of the paper, we study

the amenability constant of $Z\ell^1(G)$ for certain families of finite group G . For such G , we are able to obtain an explicit formula for $\text{AM}(Z\ell^1(G))$ that is more tractable than the corresponding expression in [1]; we apply these formulas to several well-known examples of metabelian groups, and make some comments on the values that $\text{AM}(Z\ell^1(G))$ can assume.

2 NOTATION AND OTHER PRELIMINARIES

2.1 IMPORTANT CONVENTIONS

Since we are interested in discrete groups, we will always equip finite groups with counting measure, not the uniform probability measure. Occasionally this will mean that in quoting results concerning harmonic analysis on compact groups, we have to insert a scaling factor. It should be clear, from context, when and how this is done.

INFINITE PRODUCTS AND SUMS OVER ARBITRARY INDEXING SETS. In several places we will want to consider certain infinite sums or products where the numbers involved are indexed by some arbitrary set \mathbf{I} . Since we only need to consider sums of positive numbers and products of numbers ≥ 1 , we shall interpret this as unconditional summation

$$\sum_{i \in \mathbf{I}} a_i := \sup \left\{ \sum_{i \in F} a_i : F \subseteq \mathbf{I}, |F| < \infty \right\}$$

when $a_i \geq 0$ for all $i \in \mathbf{I}$; and likewise we define

$$\prod_{i \in \mathbf{I}} b_i := \sup \left\{ \prod_{i \in F} b_i : F \subseteq \mathbf{I}, |F| < \infty \right\}$$

when $b_i \geq 1$ for all i .

Algebra characters and group characters: The Gelfand spectrum of a commutative, unital Banach algebra A will be denoted by $\text{sp}(A)$; usually we shall think of it as the space of non-zero multiplicative linear functionals, rather than the maximal ideal space.

Unfortunately, the word ‘character’ is used in both Banach algebra theory and in the representation theory of finite groups, and means two slightly different things. In this article, to try and prevent ambiguity, we will always use the phrase **algebra character** to mean a character in the sense of Gelfand theory, i.e. a non-zero multiplicative linear function from a complex Banach algebra to the ground field; and we will always use the phrase **group character** to mean a character in the sense of finite group theory, i.e. the trace of a finite-dimensional (unitary) representation.

Given a group character χ , we say that π **affords** χ when π is a finite-dimensional representation of G whose trace is χ . The **degree of** χ , which we denote by d_χ , is defined to be the dimension of any π which affords χ ; equivalently, $d_\chi = \chi(e)$ where e is the identity of the group in question.

REMARK 2.1. When viewing a group character as the trace of a suitable representation, we use the unnormalized trace, denoted by Tr , so that the trace of the $d \times d$ identity matrix is d . (In some sources, one normalizes group characters so that they each take the value 1 on the identity element of the group; our convention is more in keeping with that used in finite group theory.)

2.2 GENERAL PROPERTIES OF $Z\ell^1$

It is well known that when G is finite, algebra characters on $Z\ell^1(G)$ correspond to irreducible group characters of G . For later reference, we record a more precise version of this statement, as a lemma.

LEMMA 2.2. *Let G be a finite group. If ψ is an algebra character on $Z\ell^1(G)$, then there is a unique irreducible group character χ of G which satisfies*

$$\psi(f) = \sum_{x \in G} f(x) d_\chi^{-1} \chi(x^{-1}) \quad \text{for all } f \in Z\ell^1(G). \quad (2.1)$$

Conversely, for each irreducible group character χ on G , the formula (2.1) defines an algebra character on $Z\ell^1(G)$.

Sketch of proof. By standard properties of the nonabelian Fourier transform for finite groups, $Z\ell^1(G)$ is isomorphic as a Banach algebra to $\mathbb{C}^{\hat{G}}$. Hence it is spanned by its minimal idempotents, which are all of the form $|G|^{-1} d_\chi \chi$ for some group character χ .

Suppose ψ is an algebra character on $Z\ell^1(G)$. If we write $f = \sum_\sigma a_\sigma |G|^{-1} d_\sigma \sigma$ for appropriate complex numbers a_σ , then there exists a unique χ such that $\psi(f) = a_\chi$. By Schur orthogonality,

$$a_\chi = \sum_{x \in G} f(x) d_\chi^{-1} \chi(x^{-1}),$$

as required. Conversely, if χ is a given irreducible group character on G , then it is easily checked that defining ψ by (2.1) gives an algebra character on $Z\ell^1(G)$ \square

At certain points below we will rely crucially on a result of Rider concerning norms of central idempotents in the group algebra of a compact group. (For our applications, we only need the case of finite groups; but this restriction does not seem to make the result significantly easier to prove.)

THEOREM 2.3 (Rider; see [13, Lemma 5.2]). *Let G be a compact group, λ a Haar measure on it, and ψ a finite linear combination of irreducible group characters on G . Suppose that $\psi * \psi = \psi$ and that*

$$\int_G |\psi(x)| d\lambda(x) > 1.$$

Then

$$\int_G |\psi(x)| d\lambda(x) \geq \frac{301}{300}.$$

REMARK 2.4. Rider's result is stated for the case where $\lambda(G) = 1$. However, a simple rescaling argument shows that this is equivalent to the formulation we have given.

The following lemma will be used later, in several places.

LEMMA 2.5. *Let H and K be (discrete) FC-groups. Then the canonical, isometric isomorphism of Banach algebras*

$$\ell^1(H) \widehat{\otimes} \ell^1(K) \cong \ell^1(H \times K)$$

restricts to an isometric isomorphism of Banach algebras

$$Z\ell^1(H) \widehat{\otimes} Z\ell^1(K) \cong Z\ell^1(H \times K)$$

Proof. For any FC group H , we can define an averaging operator $P_H : \ell^1(H) \rightarrow Z\ell^1(H)$ by

$$P_H(f)(x) = \frac{1}{|C_x|} \sum_{t \in C_x} f(t)$$

where C_x denotes the conjugacy class of x in H . It is clear that P_H is well-defined (since H is an FC-group) and that P_H leaves elements of $Z\ell^1(H)$ fixed.

Now let H and K be FC groups, and let $\theta : \ell^1(H) \widehat{\otimes} \ell^1(K) \rightarrow \ell^1(H \times K)$ be the canonical isometrical isomorphism of Banach algebras, which satisfies $\theta(f \otimes g)(x, y) = f(x)g(y)$ for all $f \in \ell^1(H)$, $g \in \ell^1(K)$, $x \in H$ and $y \in K$. We claim that

$$P_{H \times K} \circ \theta = \theta \circ (P_H \otimes P_K). \quad (2.2)$$

To see this: let $f \in \ell^1(H)$ and $g \in \ell^1(K)$. If $(x, y) \in H \times K$, then since $C_{(x,y)}$ may be identified with $C_x \times C_y$, we have

$$\begin{aligned} P_{H \times K} \circ \theta(f \otimes g)(x, y) &= P_{H \times K}(f(x)g(y)) \\ &= |C_{(x,y)}|^{-1} \sum_{(t,s) \in C_x \times C_y} f(t)g(s) \\ &= |C_x|^{-1} \sum_{t \in C_x} f(t) |C_y|^{-1} \sum_{s \in C_y} g(s) \\ &= P_H(f)(x) P_K(g)(y) = \theta \circ (P_H \otimes P_K)(f \otimes g)(x, y), \end{aligned}$$

so that $P_{H \times K} \circ \theta(f \otimes g) = \theta \circ (P_H \otimes P_K)(f \otimes g)$. The claimed identity (2.2) now follows by linearity and continuity.

It is clear from (2.2) that $\theta(f \otimes g) \in Z\ell^1(H \times K)$ for all $f \in \ell^1(H)$ and $g \in \ell^1(K)$; hence $\theta(Z\ell^1(H) \widehat{\otimes} Z\ell^1(K)) \subseteq Z\ell^1(H \times K)$. To prove the converse inclusion: let $u \in Z\ell^1(H \times K)$; then $\theta^{-1}(u) \in \ell^1(H) \widehat{\otimes} \ell^1(K)$, and so

$$u = P_{H \times K} \theta(\theta^{-1}(u)) = \theta(P_H \otimes P_K)(\theta^{-1}(u)).$$

Moreover, since θ^{-1} and $P_H \otimes P_K$ both have norm 1, this shows that

$$\theta|_{Z\ell^1(H) \widehat{\otimes} Z\ell^1(K)} : Z\ell^1(H) \widehat{\otimes} Z\ell^1(K) \rightarrow Z\ell^1(H \times K)$$

is not just surjective, but is an isometry, as claimed. \square

2.3 THE RESTRICTED DIRECT PRODUCT OF FINITE GROUPS

DEFINITION 2.6. Let \mathbf{I} be an indexing set and $(G_i)_{i \in \mathbf{I}}$ a family of groups; let $\prod_{i \in \mathbf{I}} G_i$ denote the set-theoretic product of these groups, which is itself a group in a canonical way. The **restricted direct product** of the family (G_i) (shortly written as RDPF) is defined to be the subgroup

$$\bigoplus_{i \in \mathbf{I}} G_i := \left\{ (x_i)_{i \in \mathbf{I}} \in \prod_{i \in \mathbf{I}} G_i : x_i = e_{G_i} \text{ for all but finitely many } i. \right\}$$

Clearly, when the indexing set is finite, the restricted direct product agrees with the usual direct product of groups.

REMARK 2.7. Our notation for the restricted direct product of groups is borrowed from the case where each of the groups G_i is abelian: in this case, the restricted direct product is sometimes referred to as the **direct sum** (since the restricted direct product gives the coproduct in the category of Abelian groups).

PROPOSITION 2.8. *Let $(G_i)_{i \in \mathbf{I}}$ be a family of finite groups and G their restricted direct product. Then G is a FC group.*

Proof. Let $x = (x_i)_{i \in \mathbf{I}} \in G$ and let C_x denote the conjugacy class of x . Since $x_i = e_{G_i}$ for all but finitely many $i \in \mathbf{I}$, for all $y = (y_i)_{i \in \mathbf{I}}$, we have $y_i x_i y_i^{-1} = e_{G_i} \in G$ for all but finitely many $i \in \mathbf{I}$.

Define $\mathbf{I}_x := \{i \in \mathbf{I} : x_i \neq e_{G_i}\}$, which is a finite subset of \mathbf{I} . Then

$$|C_x| = \prod_{i \in \mathbf{I}_x} |C_{x_i}| \leq \prod_{i \in \mathbf{I}_x} |G_i| < \infty,$$

and since x was chosen arbitrarily, G is a FC group. \square

We are interested in examples where G is ‘not too far from abelian’ in some sense, yet $Z\ell^1(G)$ fails to have properties associated with amenability. In particular, some of our examples will be metabelian.

REMARK 2.9 (Some basic properties/observations). Let $(G_i)_{i \in \mathbf{I}}$ be a family of finite groups and let $G = \bigoplus_{i \in \mathbf{I}} G_i$.

- (i) $Z(G) = \bigoplus_{i \in \mathbf{I}} Z(G_i)$.
- (ii) $G' = \bigoplus_{i \in \mathbf{I}} G'_i$.
- (iii) If each G_i is nilpotent of class n , then so is G .
- (iv) If each G_i is solvable of length n , then so is G .

Inclusion and projection homomorphisms: Let (G_i) be a family of discrete groups, and let $F \subset \mathbf{I}$; write F^c for $\mathbf{I} \setminus F$. Since

$$\bigoplus_{i \in \mathbf{I}} G_i \cong \left(\bigoplus_{i \in F} G_i \right) \times \left(\bigoplus_{i \in F^c} G_i \right)$$

by Lemma 2.5 we obtain an isometric isomorphism of Banach algebras

$$\mathcal{Z}\ell^1\left(\bigoplus_{i \in \mathbf{I}} G_i\right) \cong \mathcal{Z}\ell^1\left(\bigoplus_{i \in F} G_i\right) \widehat{\otimes} \mathcal{Z}\ell^1\left(\bigoplus_{i \in F^c} G_i\right). \quad (2.3)$$

Hence, if we write E_F^c for the identity element of $\mathcal{Z}\ell^1(\bigoplus_{i \in F^c} G_i)$, there is a unital, isometric, homomorphism of Banach algebras

$$\iota_F : \mathcal{Z}\ell^1\left(\bigoplus_{i \in F} G_i\right) \rightarrow \mathcal{Z}\ell^1\left(\bigoplus_{i \in \mathbf{I}} G_i\right) \quad (2.4)$$

defined by $\iota_F(a) = a \otimes E_F^c$. When F is a singleton, say $\{j\}$, we denote ι_F by ι_j . If we denote by ε_{F^c} the augmentation character on $\mathcal{Z}\ell^1(\bigoplus_{i \in F^c} G_i)$, and denote by id_F the identity homomorphism on $\mathcal{Z}\ell^1(\bigoplus_{i \in F} G_i)$, then there is a unital, surjective homomorphism of Banach algebras

$$\mathbf{P}_F = \text{id}_F \otimes \varepsilon_{F^c} : \mathcal{Z}\ell^1\left(\bigoplus_{i \in \mathbf{I}} G_i\right) \rightarrow \mathcal{Z}\ell^1\left(\bigoplus_{i \in F} G_i\right) \quad (2.5)$$

which satisfies $\mathbf{P}_F(a \otimes b) = \varepsilon_{F^c}(b)a$ for all $a \in \mathcal{Z}\ell^1(\bigoplus_{i \in F} G_i)$ and all $b \in \mathcal{Z}\ell^1(\bigoplus_{i \in F^c} G_i)$.

The Gelfand spectrum: Let G be a RDPF group. Since $\mathcal{Z}\ell^1(G)$ is a commutative Banach algebra, it is natural to ask for a description of its Gelfand spectrum. This is given by the following result (recall that Lemma 2.2 gives a description of the spectrum when G is finite).

THEOREM 2.10. *Let $(G_i)_{i \in \mathbf{I}}$ be a family of discrete groups and G their restricted direct product. Then there is a homeomorphism from $\text{sp}(\mathcal{Z}\ell^1(G))$ onto*

$$\prod_{i \in \mathbf{I}} \text{sp}(\mathcal{Z}\ell^1(G_i)) := \{(\psi_i)_{i \in \mathbf{I}} : \psi_i \in \text{sp}(\mathcal{Z}\ell^1(G_i)) \forall i \in \mathbf{I}\}$$

equipped with the product topology. In particular, $\text{sp}(\mathcal{Z}\ell^1(G))$ is totally disconnected.

Proof. For each $\omega \in \text{sp}(\mathcal{Z}\ell^1(G))$ we can define $\psi_i := \omega \circ \iota_i$ for each $i \in \mathbf{I}$. ψ_i is an algebra character for $\mathcal{Z}\ell^1(G_i)$ for each $i \in \mathbf{I}$. Conversely, let

$$Zc_c(G) = \{f \in c_c(G) : f \text{ is constant on the conjugacy classes of } G\}.$$

For each set $(\psi_i)_{i \in \mathbf{I}}$, we define ω on $Zc_c(G)$ as follows: given $f \in Zc_c(G)$, choose $F \subseteq \mathbf{I}$ such that $\text{supp}(f) \subseteq \bigoplus_{i \in F} G_i \times E_F^c$. It is clear that

$$\omega_F(x) := \prod_{i \in F} \psi_i(x_i), \quad x = (x_i)_{i \in F} \in \bigoplus_{i \in F} G_i$$

will define a character group for finite group $\bigoplus_{i \in F} G_i$. Let $\omega(f) := \omega_F(\mathbf{P}_F(f))$ (\mathbf{P}_F was defined in (2.5)); therefore, ω is well-defined. Since ω is a bounded linear map so we can extend it to $Z\ell^1(G)$. On the other hand, for all $f, g \in Zc_c(G)$ we have $\omega(f * g) = \omega(f)\omega(g)$ and so ω belongs to $\text{sp}(Z\ell^1(G))$.

Let $\mathbf{j} : \text{sp}(Z\ell^1(G)) \rightarrow \prod_{i \in \mathbf{I}} \text{sp}(Z\ell^1(G_i))$ be the map defined by

$$\mathbf{j}(\omega) = (\omega \circ \iota_i)_{i \in \mathbf{I}}.$$

We have just seen that \mathbf{j} is a bijection. It remains only to show that \mathbf{j} is continuous. Let $U = \prod_{i \in \mathbf{I}} U_i \subseteq \prod_{i \in \mathbf{I}} \text{sp}(Z\ell^1(G_i))$ be a sub-basic open set i.e $U_i = \text{sp}(Z\ell^1(G_i))$ for all $i \in \mathbf{I}$ but one i_0 when

$$U_{i_0} = \{\psi \in \text{sp}(Z\ell^1(G_{i_0})) \text{ so that } |\langle \psi - \phi, f \rangle| < \varepsilon\}$$

for some $f \in Z\ell^1(G_{i_0})$ and $\phi \in \text{sp}(Z\ell^1(G_{i_0}))$. We show that $\mathbf{j}^{-1}(U)$ is open, by showing that each $\omega \in \mathbf{j}^{-1}(U)$ has a Gelfand-open neighbourhood contained in $\mathbf{j}^{-1}(U)$. Since $\omega \in \mathbf{j}^{-1}(U)$, $\delta := \varepsilon - |\langle \omega \circ \iota_{i_0} - \phi, f \rangle| > 0$. Define

$$V := \{\omega' \in \text{sp}(Z\ell^1(G)) \text{ so that } |\langle \omega' - \omega, \iota_{i_0}(f) \rangle| < \delta\}$$

an open neighborhood of ω in Gelfand topology on $\text{sp}(Z\ell^1(G))$. Clearly, for each $\omega' \in V$, $\omega' \circ \iota_{i_0} \in U_{i_0}$; therefore, $\mathbf{j}(\omega') \in U$. Since i_0 is arbitrary and f is an arbitrary element in $Z\ell^1(G_{i_0})$, Thus \mathbf{j} is continuous; since it is bijective from a compact space onto a Hausdorff one, we conclude that \mathbf{j}^{-1} is also continuous. \square

3 CHARACTERIZING ZL-AMENABILITY OF RDPF GROUPS

We can quantify amenability of a Banach algebra via its *amenability constant*, as defined in [5]. If \mathcal{A} is a Banach algebra, the *amenability constant of \mathcal{A}* , which we denote by $\text{AM}(\mathcal{A})$, is

$$\inf \left\{ \sup_{\alpha} \|\mu_{\alpha}\| : (\mu_{\alpha})_{\alpha} \text{ is a bounded approximate diagonal for } \mathcal{A} \right\}$$

where we define the infimum of the empty set to be $+\infty$. Note that if \mathcal{A} is unital, then $\text{AM}(\mathcal{A}) \geq 1$.

REMARK 3.1. Let \mathcal{A} and \mathcal{B} be Banach algebras, and let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a continuous homomorphism with dense range. It is well known that if \mathcal{A} is amenable then so is \mathcal{B} ; moreover, the proof shows that $\text{AM}(\mathcal{B}) \leq \|\phi\|^2 \text{AM}(\mathcal{A})$.

DEFINITION 3.2. Let G be a discrete group. The *derived subgroup of G* (also called the *commutator subgroup of G*) is the normal subgroup of G generated by the set of all commutators of elements in G . We denote the derived subgroup of G by G' .

It has been conjectured that, for each discrete FC group G , $Z\ell^1(G)$ is amenable if and only if G' is finite. The following result confirms this conjecture in the special case where G is an RDPF group.

THEOREM 3.3. *Let $(G_i)_{i \in \mathbf{I}}$ be a family of finite groups and let $G = \bigoplus_{i \in \mathbf{I}} G_i$. Then TFAE:*

- (i) $Z\ell^1(G)$ is amenable;
- (ii) G_i is abelian for all but finitely many i ;
- (iii) G is isomorphic to the product of a finite group with an abelian group;
- (iv) the derived subgroup of G is finite.

Proof. We start by defining $N = \{i \in \mathbf{I} : G_i \text{ is non-abelian}\}$.

(i) \implies (ii). As observed in [1], it follows from Rider's result (Theorem 2.3) that $\text{AM}(Z\ell^1(H)) > 1 + 1/300$ whenever H is a finite nonabelian group. Now suppose $Z\ell^1(G)$ is amenable, and let F be a finite subset of N . Recall that we have a quotient homomorphism of Banach algebras $\mathbf{P}_F : Z\ell^1(G) \rightarrow Z\ell^1(\bigoplus_{i \in F} G_i)$, as defined earlier in (2.5). It follows from Remark 3.1 that

$$\text{AM}(Z\ell^1(G)) \geq \text{AM}(Z\ell^1(\bigoplus_{i \in F} G_i)).$$

Moreover, it was proved in [1] that

$$\text{AM}(Z\ell^1(\bigoplus_{i \in F} G_i)) = \prod_{i \in F} \text{AM}(Z\ell^1(G_i))$$

Hence

$$\text{AM}(Z\ell^1(G)) \geq \prod_{i \in F} \text{AM}(Z\ell^1(G_i)) \geq (1 + 1/300)^{|F|}.$$

Since F was an arbitrary finite subset of N , this shows N is finite.

(ii) \implies (iii). This is clear.

(iii) \implies (i). If K is finite and A is abelian, then by Lemma 2.5,

$$Z\ell^1(K \times A) \cong Z\ell^1(K) \widehat{\otimes} Z\ell^1(A) = Z\ell^1(K) \widehat{\otimes} \ell^1(A)$$

which is the projective tensor product of two amenable Banach algebras, hence is amenable.

(ii) \iff (iv). It is pointed out in Remark 2.9(ii) that $G' = \bigoplus_{i \in I} G'_i$. Therefore G' is finite if and only if G_i is abelian for all but finitely many i .

□

4 BOUNDED APPROXIMATE IDENTITIES IN MAXIMAL IDEALS OF $Z\ell^1(G)$

If A is an amenable, unital, commutative Banach algebra then each maximal ideal of A has a bounded approximate identity; moreover, the proof shows that the norms of these bounded approximate identities are bounded from above by $1 + \text{AM}(A)$.

When G is an FC-group, $Z\ell^1(G)$ has been studied by Stegmeir in [14]: there, an example is given which has a maximal ideal without a bounded approximate identity, and which is therefore non-amenable. Stegmeir's example (see Example 5.8 below) is, although, he does not call attention to this fact, an RDPF group. We are therefore motivated to address a more general question: if $(G_i)_{i \in \mathbf{I}}$ is a family of finite groups, G is their restricted direct product, and $\psi \in \text{sp}(Z\ell^1(G))$, when does ψ have a bounded approximate identity? This will be answered by Theorem 4.1 below.

We need some preliminary observations, which all follow from basic properties of the nonabelian Fourier transform for finite groups. Let G be a finite group. As observed in the proof of Lemma 2.2, the Gelfand transform maps $Z\ell^1(G)$ isomorphically onto $\mathbb{C}^{\widehat{G}}$, and an explicit formula for the inverse of the Gelfand transform is

$$(a_\pi)_{\pi \in \widehat{G}} \mapsto \frac{1}{|G|} \sum_{\pi \in \widehat{G}} a_\pi d_\pi \text{Tr } \pi \quad (4.1)$$

Let ψ_σ be the algebra character on $Z\ell^1(G)$ that corresponds to the irreducible group representation σ . The Gelfand transform maps $\text{Ker } \psi_\sigma$ bijectively onto $\{(a_\pi)_{\pi \in \widehat{G}} : a_\sigma = 0\}$. Since the latter has an obvious identity element, namely $(a_\pi)_{\pi \in \widehat{G}}$ where $a_\pi = 0$ if $\pi = \sigma$ and 1 otherwise, we see that $\text{Ker } \psi_\sigma$ has an identity element. Denoting this identity element by u_σ , we have

$$\delta_e - u_\sigma = \frac{1}{|G|} d_\sigma \text{Tr } \sigma \quad (4.2)$$

Now suppose $G = G_1 \times \cdots \times G_n$, in which case we can identify $\ell^1(G)$ (via an isometric isomorphism of Banach algebras) with the n -fold projective tensor product $\ell^1(G_1) \widehat{\otimes} \cdots \widehat{\otimes} \ell^1(G_n)$. Let $\sigma \in \widehat{G}$: then for $i = 1, \dots, n$ there exists $\sigma_i \in \widehat{G}_i$ such that $\sigma = \sigma_1 \times \cdots \times \sigma_n$; and since $d_\sigma = \prod_{i=1}^n d_{\sigma_i}$, Equation (4.2) becomes

$$\delta_e - u_\sigma = \left(\frac{1}{|G_1|} d_{\sigma_1} \text{Tr } \sigma_1 \right) \otimes \cdots \otimes \left(\frac{1}{|G_n|} d_{\sigma_n} \text{Tr } \sigma_n \right) \in \ell^1(G_1) \widehat{\otimes} \cdots \widehat{\otimes} \ell^1(G_n). \quad (4.3)$$

THEOREM 4.1. *Let $(G_i)_{i \in \mathbf{I}}$ be a family of finite groups, and let $G = \bigoplus_{i \in \mathbf{I}} G_i$. Let $\omega \in \text{sp}(Z\ell^1(G))$, and let $(\chi_i)_{i \in \mathbf{I}}$ be the corresponding family of unnormalized characters. Then $\text{Ker } \omega$ has a bounded approximate identity, if and only if $d_{\chi_i} \|\chi_i\|_1 = |G_i|$ for all but finitely many $i \in \mathbf{I}$.*

Proof. We start by setting up some temporary notation. Define for each $F \subseteq \mathbf{I}$ finite, $H_F = \bigoplus_{i \in F} G_i$ and $H_F^c = \bigoplus_{i \in \mathbf{I} \setminus F} G_i$, so that $Z\ell^1(G) = Z\ell^1(H_F \times H_F^c)$. Write E_F^c for the identity element of $Z\ell^1(H_F^c)$, i.e. the unit point mass concentrated at the identity element of H_F^c , and likewise write E_F for the identity element of $Z\ell^1(H_F)$. We also write d_i for the degree of χ_i .

First suppose that $d_i \|\chi_i\|_1 = |G_i|$ for all but finitely many $i \in \mathbf{I}$. As in Equation (4.3), define $u_F \in Z\ell^1(H_F)$ by

$$u_F = E_F - \bigotimes_{i \in F} \frac{1}{|G_i|} d_i \chi_i.$$

Let $\omega_F := \omega \iota_F$, where $\iota_F : Z\ell^1(H_F) \rightarrow Z\ell^1(G)$ was defined in (2.4). Then u_F is the identity element of $\text{Ker } \omega_F$. As $\widehat{\otimes}$ is a cross-norm,

$$\sup_F \|u_F\|_1 \leq 1 + \sup_F \left\| \bigotimes_{i \in F} \frac{1}{|G_i|} d_i \chi_i \right\| = 1 + \sup_F \prod_{i \in F} \left(\frac{1}{|G_i|} d_i \|\chi_i\|_1 \right) < \infty.$$

Moreover, since $\|\iota_F(u_F)\|_1 = \|u_F\|_1$, the family $(\iota_F(u_F))_{F \subseteq \mathbf{I}, |F| < \infty}$ is bounded. We claim that it is, when ordered by inclusion of finite subsets, a bounded approximate identity for $\text{Ker } \omega$. To prove this, it will suffice to prove that

$$\lim_{F \subseteq \mathbf{I}, |F| < \infty} (\iota_F(u_F)) * f = f \quad \text{for all } f \in \text{Ker } \omega; \quad (4.4)$$

and by a standard approximation argument, we may assume without loss of generality that f has compact (i.e. finite) support. Thus, given $f \in Zc_c(G) \cap \text{Ker } \omega$, let S be the support of f ; if F is any finite subset of \mathbf{I} containing S , then

$$f = \iota_S \mathbf{P}_S(f) = \iota_F \mathbf{P}_F(f)$$

(where \mathbf{P}_S and \mathbf{P}_F are the homomorphisms defined in (2.5)). Moreover, $0 = \omega(f) = \omega_F(\mathbf{P}_F(f))$. Consequently $\mathbf{P}_F(f) \in \text{Ker } \omega_F$, and thus

$$f * \iota_F(u_F) = \iota_F(\mathbf{P}_F(f) * u_F) = \iota_F(\mathbf{P}_F(f)) = f$$

for all finite F containing S . This proves Equation (4.4), as required.

Conversely, suppose that $\text{Ker } \omega$ has a bounded approximate identity say $(h_\alpha)_\alpha$. For each $F \subseteq \mathbf{I}$ when $|F| < \infty$ define

$$\Lambda_F^\omega(f \otimes g) = \omega_{F^c}(g)f \quad \text{for all } f \in Z\ell^1(H_F) \text{ and } g \in Z\ell^1(H_F^c).$$

Since $\omega_{F^c} = \omega \iota_{F^c}$ has norm 1, being an algebra character, we have

$$\|\Lambda_F^\omega(f \otimes g)\|_1 \leq \|f\|_1 \|g\|_1,$$

and so Λ_F^ω defines a linear contraction from $Z\ell^1(H_F \times H_F^c)$ onto $Z\ell^1(H_F)$, using Lemma 2.5.

Moreover, given $f_1, f_2 \in Z\ell^1(H_F)$ and $g_1, g_2 \in Z\ell^1(H_F^c)$, it is easily checked that

$$\Lambda_F^\omega((f_1 \otimes g_1) * (f_2 \otimes g_2)) = \Lambda_F^\omega(f_1 \otimes g_1) * \Lambda_F^\omega(f_2 \otimes g_2);$$

hence, by linearity and continuity, Λ_F^ω is an algebra homomorphism.

Observe, since $\omega_F \Lambda_F^\omega = \omega$, that $\Lambda_F^\omega(\text{Ker } \omega) \subseteq \text{Ker } \omega_F$. Moreover, for each $f \in \text{Ker } \omega_F$, $\iota_F(f) \in \text{Ker } \omega$, and $\Lambda_F^\omega \iota_F(f) = f$. Since $u_F \in \text{Ker } \omega_F$ and (h_α) is a bounded approximate identity for $\text{Ker } \omega_F$,

$$\begin{aligned} \|\Lambda_F^\omega(h_\alpha) - u_F\|_1 &= \|\Lambda_F^\omega(h_\alpha) * u_F - u_F\|_1 \\ &= \|\Lambda_F^\omega(h_\alpha) * \Lambda_F^\omega \iota_F(u_F) - \Lambda_F^\omega \iota_F(u_F)\|_1 \\ &= \|\Lambda_F^\omega(h_\alpha * \iota_F(u_F) - \iota_F(u_F))\|_1 \rightarrow 0. \end{aligned}$$

Because $\|\Lambda_F^\omega\| \leq 1$,

$$\sup_{F \subseteq \mathbf{I}, |F| < \infty} \sup_{\alpha} \|\Lambda_F^\omega(h_\alpha)\|_1 \leq \sup_{\alpha} \|h_\alpha\|_1 \leq M$$

for some $M > 0$, thus $\|u_F\|_1 \leq M$ for all finite subsets $F \subseteq \mathbf{I}$. Hence

$$\prod_{i \in F} \frac{d_i}{|G_i|} \|\chi_i\|_1 = \|E_F - u_F\|_1 \leq M + 1 \quad (4.5)$$

Let $i \in \mathbf{I}$. For each i , $|G_i|^{-1}d_i\chi_i$ is a central idempotent in the group algebra $\ell^1(G_i)$; in particular it has ℓ^1 -norm ≥ 1 . Moreover, by Rider's theorem (Theorem 2.3),

$$\text{either } |G_i|^{-1}d_i\|\chi_i\|_1 = 1 \quad \text{or} \quad |G_i|^{-1}d_i\|\chi_i\|_1 \geq \frac{301}{300}.$$

It therefore follows from (4.5) that

$$|\{i \in \mathbf{I} : d_i\|\chi_i\|_1 > |G_i|\}| \leq \frac{\log(M+1)}{\log 301 - \log 300} < \infty.$$

In particular, $d_i\|\chi_i\|_1 = |G_i|$ for all but finitely many i . \square

THEOREM 4.2. *Let $(G_i)_{i \in \mathbf{I}}$ be a family of finite groups and let $G = \bigoplus_{i \in \mathbf{I}} G_i$. Then the following are equivalent:*

- (i) *every maximal ideal in $Z\ell^1(G)$ has a bounded approximate identity;*
- (ii) *there is a finite subset $F \subset \mathbf{I}$ such that, for each $i \in \mathbf{I} \setminus F$ and each irreducible group character χ of G_i , we have $d_\chi\|\chi\|_1 = |G_n|$.*
- (iii) *there exists a constant $M > 0$ such that each maximal ideal in $Z\ell^1(G)$ has a bounded approximate identity of norm $\leq M$.*

Proof.

(iii) \implies (i). This is trivial.

(ii) \implies (iii). Let F be as assumed in (ii), and define

$$M := \prod_{i \in F} \sup_{\pi \in \widehat{G}} \frac{d_\pi}{|G_n|} \|\text{Tr } \pi\|_1 < \infty.$$

Given $\omega \in \text{sp}(Z\ell^1(G))$, let (χ_i) be the corresponding family of (irreducible) group characters, and let d_i denote the degree of χ_i . For each finite subset $T \subset \mathbf{I}$, define $u_T \in Z\ell^1(\bigoplus_{i \in T} G_i) = \widehat{\bigotimes_{i \in T} Z\ell^1(G_i)}$ by

$$u_T = \delta_e - \bigotimes_{i \in T} \frac{d_i}{|G_i|} \chi_i$$

Order the net $(\iota_T(u_T))$, where T ranges over all finite subsets of \mathbf{I} , by inclusion. Then by an argument like that in the proof of Theorem 4.1, $(\iota_T(u_T))$ is a bounded approximate identity for $\text{Ker } \omega$, with $\sup_T \|\iota_T(u_T)\| \leq M + 1$.

(i) \implies (ii). We prove the contrapositive. Suppose that (ii) does not hold. Then there exists an infinite set $S \subset \mathbf{I}$, and for each $j \in S$ an irreducible group character ϕ_j on G_j , such that $d_j \|\phi_j\|_1 \neq |G_j|$. Since $|G_j|^{-1} d_j \phi_j$ is an idempotent in $Z\ell^1(G_j)$, the result of Rider implies that $|G_j| d_j \|\phi_j\|_1 \geq 301/300$. Now let ω in $\text{sp}(Z\ell^1(G))$ be such that the corresponding family (χ_i) of group characters satisfies $\chi_j = \phi_j$ for all $j \in \mathbf{I}$. Then as in the proof of Theorem 4.1, we can show that $\text{Ker}(\omega)$ does not have a bounded approximate identity. \square

We have already observed that, for any irreducible group character χ on a finite group G , we have the lower bound $d_\chi \|\chi\|_1 \geq |G|$. Let us examine this inequality more closely.

LEMMA 4.3. *Let G be a finite group and χ an irreducible character on G . Then $d_\chi \|\chi\|_1 \geq |G|$. Moreover, equality holds if and only if*

$$|\chi(x)| \in \{0, d_\chi\} \quad \text{for all } x \in G. \quad (4.6)$$

Proof. Since χ is irreducible, $\sum_{x \in G} |\chi(x)|^2 = |G|$. Hence the first statement in the lemma follows from the trivial inequality

$$\sum_{x \in G} |\chi(x)|^2 \leq d_\chi \sum_{x \in G} |\chi(x)|. \quad (*)$$

For the second statement, we need to show that equality holds in $(*)$ if and only if (4.6) is satisfied. The ‘if’ direction is clear. Conversely, if (4.6) is not satisfied, pick $y \in G$ such that $0 < |\chi(y)| < d_\chi$. Then $|\chi(y)|^2 < d_\chi |\chi(y)|$, so that

$$\sum_{x \in G} |\chi(x)|^2 = |\chi(y)|^2 + \sum_{x \in G \setminus \{y\}} |\chi(x)|^2 < d_\chi |\chi(y)| + d_\chi \sum_{x \in G \setminus \{y\}} |\chi(x)| = d_\chi \sum_{x \in G} |\chi(x)|$$

as required. \square

Condition (4.6) is very restrictive, and is related to some concepts from the character theory of finite groups which we now briefly describe.

DEFINITION 4.4 (The centre of a group character). Let χ be an irreducible group character on a finite group G , of degree d say. Let π be any representation which affords χ , then

$$\{x \in G: |\chi(x)| = d\} = \{x \in G: \pi(x) \in \mathbb{C}I_d\} \supseteq Z(G)$$

Thus the set $\{x \in G: |\chi(x)| = d_\chi\}$ is a normal subgroup of G , usually denoted by $\mathbf{Z}(\chi)$ and called the **centre of χ** .

Condition (4.6) therefore says that χ is induced from a linear character on its centre. For sake of convenience we shall adopt some non-standard terminology, and say that a character satisfying (4.6) is **1-minimal**; the terminology is motivated by the idea that for such characters the lower bound on the ℓ^1 -norm is attained. Clearly each linear character is 1-minimal. For sake of brevity, we shall adopt the terminology that a finite group G is **1-minimal** if each irreducible character of G is 1-minimal. It follows easily from the

	$\{1\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$	$\{r, r^3\}$	$\{r^2\}$
χ_0	1	1	1	1	1
χ_1	1	-1	-1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	-1	1	-1	1
χ_σ	2	0	0	0	-2

Figure 1: Character table of $D_4 = \langle r, s \mid sr = r^3s \rangle$

definition that quotients of 1-minimal groups and products of 1-minimal groups are also 1-minimal .

Discussion of hereditary properties implicitly presupposes that we can find some non-abelian examples (otherwise our discussion would be rather vacuous). The smallest such example is the dihedral group of order 8, whose character table is shown in Figure 1. This group turns out to fit into a whole family of examples, which we shall now describe.

EXAMPLE 4.5 (Extraspecial p -groups). Fix a prime p . A finite group G is **p -extraspecial** if it has order p^{2n+1} for some integer n and has the following properties:

- (i) The centre $Z(G)$ is cyclic subgroup of order p .
- (ii) The derived subgroup, G' , has order p .
- (iii) The quotient $G/Z(G)$ is abelian, and each non-identity element in the quotient has order p .

Such groups do exist (for instance, the dihedral group of order 8 is 2-extraspecial), and their character tables and conjugacy classes turn out to be uniquely determined by these conditions. In particular, if G is extraspecial of order p^{2n+1} , then each non-linear irreducible group character of G is supported on $Z(G)$ and has degree p^n . It follows from this that every irreducible group character of G is 1-minimal .

The authors are unaware of any intrinsic characterization of non-abelian 1-minimal groups, although the evidence so far suggests that such groups are fairly rare. We can at least rule out many groups using the following result, which was communicated to the second author by F. Ladisch.

THEOREM 4.6 (Ladisch, [7]). *Let G be a finite group which is 1-minimal. Then G is nilpotent.*

The proof relies on some standard arguments from the character theory of finite groups; for sake of interest and completeness, we have included a proof in Appendix A.

5 c_0 AND ℓ^p ESTIMATES FOR ELEMENTS OF $\text{sp}(\text{Z}\ell^1(G))$

5.1 PRELIMINARY DISCUSSIONS

Consider a discrete FC group G . Given an algebra character φ on $\text{Z}\ell^1(G)$, one obtains a positive definite (class) function $\tilde{\varphi} : G \rightarrow \mathbb{C}$ by

$$\tilde{\varphi}(f) = \sum_{x \in G} f(x) \varphi(x),$$

which satisfies $\tilde{\varphi}(e_G) = 1$ and $\tilde{\varphi}(y^{-1}xy) = \tilde{\varphi}(x)$ for all $x, y \in G$. Conversely, if $ZP_1(G)$ denotes the set of positive definite functions from G into \mathbb{C} which take the value 1 at e_G and are constant on the conjugacy classes of G , then the extreme points of $ZP_1(G)$ are precisely those functions $\tilde{\varphi}$ that arise by the previous formula. (For a proof, see [10, Theorem 2.7], which proves the corresponding statement for a more general class of locally compact groups.) We shall therefore abuse terminology, and say that an algebra character φ **belongs to c_0** , or **to ℓ^p** for some $1 \leq p < \infty$, when the corresponding function $\tilde{\varphi}$ is in $c_0(G)$ or $\ell^p(G)$ respectively.

When $G = \bigoplus_{i \in \mathbf{I}} G_i$ is an RDPF group, we can make this correspondence between algebra characters on $\text{Z}\ell^1(G)$ and certain functions on G completely explicit. Namely, given $\omega \in \text{sp}(\text{Z}\ell^1(G))$, let $(\chi_i)_{i \in \mathbf{I}}$ be the corresponding family of irreducible group characters, and let ψ be the function on G corresponding to ω ; then for each $x = (x_i)_{i \in \mathbf{I}} \in G$ we have

$$\tilde{\omega}(x) = \prod_{i \in \mathbf{I}} d_i^{-1} \chi_i(x_i) \quad (5.1)$$

where d_i is the degree of χ_i for each $i \in \mathbf{I}$. (The product is well-defined, because $d_i^{-1} \chi_i(x_i) = d_i^{-1} \chi_i(e_{G_i}) = 1$ for all but finitely many $i \in \mathbf{I}$.) Note that since the family $(d_i^{-1} \chi_i)_{i \in \mathbf{I}}$ satisfies the conditions of Lemma 5.11, we can use that lemma to calculate $\|\tilde{\omega}\|_p$ in terms of the group characters (χ_i) .

We can now state the following result of Stegmeir.

LEMMA 5.1 (Stegmeir, [14, Lemma 3]). *Let G be a discrete FC group G and let ω be an element of $\text{sp}(\text{Z}\ell^1(G))$. Suppose that $\text{Ker } \omega$ has a bounded approximate identity and $\omega \in c_0$. Then $\omega \in \ell^2$.*

The proof in [14] is a somewhat indirect argument by contradiction, using the existence and basic properties of a Plancherel measure on the maximal ideal space of $\text{Z}\ell^1(G)$. One motivation for the present work was to obtain a more concrete approach in the more restricted setting of restricted direct products of finite groups.

Stegmeir originally applied Lemma 5.1 to show that for a certain FC group G , constructed explicitly in [14], $\text{Z}\ell^1(G)$ has a maximal ideal without a bounded approximate identity. We will present his example later, and – since it is in fact an RDPF group – we will use Theorem 4.1 to obtain a direct proof which does not require Lemma 5.1. (En route, we will see that for RDPF groups, Lemma 5.1 is in fact not very useful, see Theorem 5.9 and the remarks preceding it.)

In view of Lemma 5.1, it is natural to attempt to classify the characters on $\text{Z}\ell^1(G)$ which lie in $c_0(G)$, and those which lie in $\ell^2(G)$. We shall provide partial results in this direction, restricting attention to the case where G is an RDPF group.

5.2 CHARACTERIZING WHEN $\omega \in c_0$

DEFINITION 5.2. Let χ be a group character on a finite group G . The **maximal character ratio** of χ , denoted by $\text{mcr}(\chi)$, is defined to be

$$\sup\{d_\chi^{-1}|\chi(g)| : g \in G \setminus \{e\}\}$$

and is clearly a real number in $[0, 1]$.

REMARK 5.3. We note some basic properties of the maximal character ratio, whose proofs are easy and are therefore omitted.

- (i) If χ is irreducible, then $\text{mcr}(\chi) > 0$.
- (ii) Let χ be a group character on H and ψ a group character on K . Then $\chi \otimes \psi$ is a group character on $H \times K$, and $\text{mcr}(\chi \otimes \psi) = \max(\text{mcr}(\chi), \text{mcr}(\psi))$.

THEOREM 5.4. Let $G = \bigoplus_{i \in \mathbf{I}} G_i$, let $\omega \in \text{sp}(\text{Z}\ell^1(G))$, and let (χ_i) be the corresponding family of group characters. Then $\omega \in c_0$ if and only if $(\text{mcr}(\chi_i))_{i \in \mathbf{I}} \in c_0(\mathbf{I})$.

Proof. Suppose that $\omega \in c_0$. Let $\varepsilon > 0$, and choose a finite subset $S \subseteq G$ such that $|\tilde{\omega}(x)| < \varepsilon$ for all $x \in G \setminus S$. Let $F \subseteq \mathbf{I}$ be a finite subset satisfying

$$S \subseteq \bigoplus_{i \in F} G_i \times \bigoplus_{i \in \mathbf{I} \setminus F} \{e_{G_i}\}.$$

Let $j \in \mathbf{I} \setminus F$; then for each $y \in G_j \setminus \{e_{G_j}\}$ we have $d_j^{-1}|\chi_j(y)| = |\tilde{\omega}(\iota_j(y))| < \varepsilon$, so that $\text{mcr}(\chi_j) < \varepsilon$. Thus $(\text{mcr}(\chi_j))_{j \in F} \in c_0$.

Conversely, suppose that $(\text{mcr}(\chi_j))_{j \in F} \in c_0$. Let $\varepsilon > 0$; then by hypothesis there exists a finite subset $F \subseteq \mathbf{I}$ such that

$$|d_i^{-1}\chi_i(y)| < \varepsilon \quad \text{for each } j \in \mathbf{I} \setminus F \text{ and each } y \in G_j \setminus \{e_{G_j}\}. \quad (5.2)$$

Define $S = \{x = (x_i) \in G : x_j = e_{G_j} \text{ for all } j \in \mathbf{I} \setminus F\}$, which is a finite subset of G . Let $x \in G \setminus S$; then there exists $j \in \mathbf{I} \setminus F$ such that $x_j \in G_j \setminus \{e_{G_j}\}$, and so by (5.2) we have

$$|\tilde{\omega}(x)| = \prod_{i \in \mathbf{I}} d_i^{-1}|\chi_i(x_i)| \leq d_j^{-1}|\chi_j(x_j)| < \varepsilon.$$

Hence $\omega \in c_0$. □

EXAMPLE 5.5. Let $G = \bigoplus_{i \in \mathbf{I}} G_i$ and let $\omega \in \text{sp} \text{Z}\ell^1(G)$, with (χ_i) being the corresponding family of group characters.

- (i) Suppose that G_i is abelian for infinitely many i . For each i such that G_i is abelian, χ_i is linear and so $\text{mcr}(\chi_i) = 1$. Therefore, by Theorem 5.4, $\omega \notin c_0$.
- (ii) Suppose there is a fixed finite non-abelian group K such that $G_i = K$ for infinitely many i . Define m to be the minimum value of $\text{mcr}(\chi)$ as χ runs over all irreducible group characters on K ; then $m > 0$ (see Remark 5.3), and $\text{mcr}(\chi_i) \geq m$ for infinitely many i . So once again, we know by Theorem 5.4 that $\omega \notin c_0$.

To obtain algebra characters on $Z\ell^1(G)$ which *do* lie in c_0 , we need to have examples of group characters whose maximal character ratio can be arbitrarily small. We shall describe two families of such examples: the first will be on a certain family of finite simple groups; the second, on a certain family of (non-nilpotent) metabelian groups. Groups in the second family will be used later as convenient building blocks for various examples.

EXAMPLE 5.6. Let $q = 2^n$ for some $n \geq 2$, and let \mathbb{F}_q be the finite field of order q . Consider $SL(2, q)$, the special linear group over \mathbb{F}_q ; this is known to be simple. It acts by projective transformations on the projective line over \mathbb{F}_q , and hence has a (transitive) permutation representation on the set $\mathbb{F}_q \cup \{\infty\}$.

The character χ of this permutation representation satisfies

$$\|\chi\|_2^2 = 2|SL(2, q)|;$$

hence, by subtracting the augmentation character from χ , we obtain an irreducible character St_q that has degree q . (St_q is called the **Steinberg character** of $SL(2, q)$.) By consulting known tables (see, e.g. [4, Ch. 28, exercise 2]), or examining the permutation representation to work out the values taken by St_q , we find that $\text{mcr}(\text{St}_q) = 1/q$.

Hence, if we take any increasing sequence of positive integers $(n_i)_{i \in \mathbb{N}}$, and let $q_i = 2^{n_i}$, $G = \bigoplus_{i \in \mathbb{N}} SL(2, q_i)$, and $\omega \in \text{sp}(Z\ell^1(G))$ the algebra character corresponding to the family $(\text{St}_{q_i})_{i \in \mathbb{N}}$, then ω lies in c_0 .

(One could replace q with any prime power ≥ 4 ; we still obtain an irreducible character St_q on $SL(2, q)$, but since $SL(2, q)$ now has non-trivial centre we will have $\text{mcr}(\text{St}) = 1$; on the other hand, St descends to a character on $PSL(2, q)$ which does have maximal character ratio $1/q$. We omit the details.)

EXAMPLE 5.7 (Affine groups of finite fields). Let \mathbb{F}_q be a finite field of order q . The **affine group of \mathbb{F}_q** , which we shall denote by $\text{Aff}(q)$, is defined to be the set

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\}$$

equipped with the group structure it inherits from the usual matrix product and inversion. It is a metabelian group; more precisely, it is isomorphic to the semidirect product $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$.

The character table of $\text{Aff}(q)$ is well known (see [15, Chapter 16, Table II.3], for instance) and is shown in Figure 2. Observe that $\text{Aff}(q)$, which has order $q(q-1)$, has $q-1$ linear characters (corresponding to those on the quotient group \mathbb{F}_q^\times), and precisely one non-linear character, which has degree $q-1$ and maximal character ratio $1/(q-1)$.

Hence, if we take any increasing sequence of prime numbers $(q_i)_{i \in \mathbb{N}}$, and let $G = \bigoplus_{i \in \mathbb{N}} \text{Aff}(q_i)$, χ_{π_i} the non-linear character on $\text{Aff}(q_i)$, and $\omega \in \text{sp}(Z\ell^1(G))$ the algebra character corresponding to the family $(\chi_{\pi_i})_{i \in \mathbb{N}}$, then ω lies in c_0 .

EXAMPLE 5.8 (Stegmeir). Let P denote the set of all prime numbers, and for each $p \in P$ define $G_p := \prod_{i=1}^{p-1} \text{Aff}(p)$. Writing π for the unique non-linear irreducible representation of $\text{Aff}(p)$, let

$$\chi_p = (\text{Tr } \pi) \otimes \cdots \otimes (\text{Tr } \pi) \quad (p \text{ times})$$

	dimension of character	$\{e\}$	C_1	$C_{(0,y)} \quad y \in 2, \dots, q-1$
size of conjugacy class		1	$q-1$	q
χ_1	1	1	1	1
$\chi_j \quad j \in 2, \dots, p-1$	1	1	1	$\theta_j((0, y))$
χ_π	$q-1$	$q-1$	-1	0

Figure 2: Character table for $\text{Aff}(q)$

which is an irreducible group character on G_p .

Let $G := \bigoplus_{p \in P} G_p$, and let $\omega \in \text{sp}(\mathbb{Z}\ell^1(G))$ be the algebra character corresponding to the family $(\chi_p)_{p \in P}$ (see Theorem 2.10). Since the group character $\text{Tr } \pi$ has maximal character ratio $(p-1)^{-1}$, so does χ_p (by Remark 5.3), for each $p \in P$. Therefore, by Theorem 5.4, ω lies in c_0 .

The original point of Stegmeir's example is that the ω defined above lies in c_0 but not in ℓ^2 ; hence, by Lemma 5.1 above, $\text{Ker } \omega$ has no bounded approximate identity. However, Lemma 5.1 turns out to be somewhat misleading when dealing with restricted direct products of finite groups. Recall that the lemma says: “if $\text{Ker } \omega$ has a bounded approximate identity and $\omega \in c_0(G)$, then $\omega \in \ell^2(G)$ ”. The next result shows that if G is an RDPF group – as in the example Stegmeir considers – then this statement is conditioning on an empty set.

THEOREM 5.9. *Let (G_i) be a family of finite groups and let $G = \bigoplus_{i \in \mathbf{I}} G_i$. Let $\omega \in \text{sp}(\mathbb{Z}\ell^1(G))$ and suppose that $\text{Ker } \omega$ has a bounded approximate identity. Then $\omega \notin c_0$.*

Proof of Theorem 5.9. Let $(\chi_i)_{i \in \mathbf{I}}$ be the family of irreducible group characters which corresponds to ω , and as usual let d_i denote the degree of χ_i . By Theorem 4.1 and Lemma 4.2, there is a finite set $F \subset \mathbf{I}$ such that for all $i \in \mathbf{I} \setminus F$, the function $|\chi_i| : G_i \rightarrow \mathbb{C}$ takes values in $\{0, d_i\}$.

Observe that for any finite group H and any irreducible character ψ on H , there is some $y \in H \setminus \{e\}$ such that $\psi(y) \neq 0$. (This is trivially true if ψ is the constant function 1; and if it is not, then Schur orthogonality gives

$$0 = \sum_{x \in G} \psi(x) = d_\psi + \sum_{x \in \text{supp}(\psi) \setminus \{e\}} \psi(x)$$

so that $\text{supp}(\psi) \setminus \{e\}$ is non-empty, as claimed.) Hence, from the previous paragraph, for each $i \in \mathbf{I} \setminus F$ there exists $y_i \in G_i \setminus \{e_{G_i}\}$ with $|\chi_i(y_i)| = d_{\chi_i}$, so that $\text{mcr}(\chi_i) = 1$. Now apply Theorem 5.4. \square

REMARK 5.10. Going back to Stegmeir's example, we can show more directly that $\text{Ker } \omega$ has no bounded approximate identity, by following the proof of Theorem 4.1 and using the information in Figure 2 to explicitly compute the ℓ^1 -norms of the group characters that make up ω . In particular, we can do without Rider's theorem (Theorem 2.3), which seems to be needed for the proof of Theorem 5.9.

5.3 EXAMPLES WHERE $\omega \in \ell^p$ FOR VARIOUS p

Although we are primarily interested in deciding whether or not $\omega \in \ell^2$, in most of our examples the calculations can be done just as easily for ℓ^p .

In the following lemma, infinite sums and products of real numbers are to be understood in the sense of Section 2.

LEMMA 5.11. *Let $(G_i)_{i \in \mathbf{I}}$ be a family of finite groups, and let $G = \bigoplus_{i \in \mathbf{I}} G_i$. We denote the identity element of each G_i by e_i .*

For each $i \in \mathbf{I}$ let $\psi_i : G_i \rightarrow \mathbb{C}$ be a function satisfying $\psi_i(e_i) = 1$, and define $\Psi : G \rightarrow \mathbb{C}$ by

$$\Psi(x) = \prod_{i \in \mathbf{I}} \psi_i(x_i) \quad \text{for } x = (x_i)_{i \in \mathbf{I}}. \quad (5.3)$$

Then for each $p \in (0, \infty)$ we have

$$\sum_{x \in G} |\Psi(x)|^p = \prod_{i \in \mathbf{I}} \|\psi_i\|_p^p. \quad (5.4)$$

In particular, if $\omega \in \text{sp}(\text{Z}\ell^1(G))$ and $(\chi_i)_{i \in \mathbf{I}}$ is the corresponding family of group characters, let d_i denote the degree of χ_i ; then

$$\|\tilde{\omega}\|_p = \prod_{i \in \mathbf{I}} d_i^{-1} \|\chi_i\|_p. \quad (5.5)$$

for every $p \in [1, \infty)$.

Proof. First of all, we note that the the product on the right hand side of (5.3) is well-defined, since $x_i = e_i$ for all but finitely many i . Also, since $\|\psi_i\|_p^p \geq \psi_i(e_i) = 1$ for all $i \in \mathbf{I}$, the infinite product on the right-hand side of (5.4) is well-defined.

For each finite subset $F \subseteq \mathbf{I}$, let

$$G_F = \{x = (x_i) \in G : x_i = e_i \text{ for all } i \in \mathbf{I} \setminus F\},$$

which is a finite subset of G . We have

$$\sum_{x \in G_F} |\Psi(x)|^p = \sum_{x \in G_F} \prod_{i \in F} |\psi_i(x_i)|^p = \prod_{i \in F} \sum_{x_i \in G_i} |\psi_i(x_i)|^p = \prod_{i \in F} \|\psi_i\|_p^p \quad (*)$$

(all sums and products being over finite indexing sets). Note also that each finite subset of G is contained in one of the form G_F for some finite subset $F \subseteq \mathbf{I}$. Hence

$$\sum_{x \in G} |\Psi(x)|^p = \sup_{F \subseteq \mathbf{I}, |F| < \infty} \sum_{x \in G_F} |\Psi(x)|^p$$

which, by (*), implies that

$$\sum_{x \in G} |\Psi(x)|^p = \sup_{F \subseteq \mathbf{I}, |F| < \infty} \prod_{i \in F} \|\psi_i\|_p^p = \prod_{i \in \mathbf{I}} \|\psi_i\|_p^p$$

which gives us the desired identity (5.4). The last part of the lemma now follows, by using the identity (5.1) and the observation that $d_i \chi_i(e_i) = 1$ for all i . \square

EXAMPLE 5.12 (An example using the Steinberg characters). Take $G = \bigoplus_{n \geq 2} SL(2, 2^n)$, and let $\omega \in \text{sp}(\mathbb{Z}\ell^1(G))$ be the algebra character corresponding to the sequence $(\text{St}_{2^n})_{n \geq 2}$, where St_{2^n} is the Steinberg character on $SL(2, 2^n)$ as defined in Example 5.6. By Theorem 5.4, $\omega \in c_0$. Consulting the known character table of $SL(2, 2^n)$, we find that

$$\|\tilde{\omega}\|_s^s = \prod_{n=2}^{\infty} \frac{2^{ns} + 2^{3n} - 2^{2n} - 2^n}{2^{ns}}.$$

From this and (5.4), we see that $\omega \in \ell^s$ if and only if $s > 3$.

In some sense, Lemma 5.11 characterizes when $\omega \in \ell^p$, in terms of the ℓ^p -norms of the irreducible group characters that make up ω . On the other hand, the ℓ^p -norms of irreducible group characters are not well understood, unless $p = 2$ or the characters are linear. In the case $p = 2$ we can be slightly more specific: the following result follows immediately from the identity (5.5), together with the standard result that the ℓ^2 -norm of an irreducible group character on a finite group H is $|H|^{1/2}$.

COROLLARY 5.13. *Let $G = \bigoplus_{i \in \mathbf{I}} G_i$ and let $\omega \in \text{sp}(\mathbb{Z}\ell^1(G))$; let $(\chi_i)_{i \in \mathbf{I}}$ be the corresponding family of (unnormalized) irreducible characters, and let d_i be the degree of χ_i . Then*

$$\|\tilde{\omega}\|_2 = \prod_{i \in \mathbf{I}} \frac{|G_i|^{1/2}}{d_i} \quad (5.6)$$

EXAMPLE 5.14 (Stegmeir's example, revisited). Let $G = \bigoplus_{p \in P} G_p$ and ω be as described in Example 5.8. Stegmeir showed in [14], by direct calculation, that $\omega \notin \ell^2$. This also follows easily from the formula (5.6): for $|G_p| = |\text{Aff}(p)|^{p-1} = p^{p-1}(p-1)^{p-1}$, while $d_p = (p-1)^{p-1}$, giving

$$\|\tilde{\omega}\|_2^2 = \prod_{p \in P} \left(1 + \frac{1}{p-1}\right)^{p-1} \geq \prod_{p \in P} 2 = +\infty$$

We can go further if we use Lemma 5.11, since by examining the character table of $\text{Aff}(q)$ we see that for $s \in [1, \infty)$,

$$(d_p^{-1} \|\chi_p\|_s)^s = (d_{\text{Tr } \pi}^{-s} \|\text{Tr } \pi\|_s^s)^{p-1} = \left(\frac{(p-1)^s + (p-1)}{(p-1)^s} \right)^{p-1} = \left(1 + \frac{1}{(p-1)^{s-1}} \right)^{p-1}.$$

It is then straightforward to show, as pointed out to the first author by G. Robinson (personal communication, MathOverflow), that $\omega \in \ell^s$ if and only if $s > 3$.

REMARK 5.15. Going back to Equation (5.6), we note that it imposes strong restrictions on the family (G_i) if we wish to obtain some ω lying in ℓ^2 . Namely, the following must hold: *for each $\delta > 0$ there exists a finite group H and an irreducible character χ on H which satisfies $(1+\delta)d_\chi^2 \geq |H|$.* In this context, we raise the following problem, which may be of interest to researchers in finite group theory: *given $\delta \in [0, 1/2)$, classify all finite groups H with the property that $(1+\delta)\chi^2 \geq |H|$ for some irreducible character χ on H .* The authors know of no value of δ for which such a classification is known.

It is natural to wonder what summability properties can be exhibited by various ω . Clearly, if G is virtually abelian then none of the algebra characters on $Z\ell^1(G)$ can lie in c_0 . The following examples will show that within the class of metabelian groups, one can obtain examples with characters lying in c_0 but not in ℓ^s for $s < \infty$, and at the other extreme obtain examples with characters lying in every ℓ^s for $s > 1$.

EXAMPLE 5.16. For each prime $p \in P$, let $G_p = \text{Aff}(p)^{2^p}$. Let $\omega \in \text{sp}(Z\ell^1(G))$, let $(\chi_p)_{p \in P}$ be the corresponding family of group characters, and let d_p denote the degree of χ_p . Note that since $\text{Aff}(p)$ has an irreducible group character with maximal character ratio $(p-1)^{-1}$, so does G_p (see Remark 5.3), and therefore it is possible to have $\omega \in c_0$.

On the other hand: let $s \in [1, \infty)$. By inspecting the possible cases, we see that

$$\left(d_p^{-1} \|\psi\|_s\right)^s \geq 1 + \frac{1}{(p-1)^{s-1}} \quad \text{for every irreducible group character } \psi \text{ on } \text{Aff}(p).$$

Therefore, since χ_p is a tensor product of irreducible characters on $\text{Aff}(p)$,

$$\left(d_p^{-1} \|\chi_p\|_s\right)^s \geq \left(1 + \frac{1}{(p-1)^{s-1}}\right)^{2^p} > 1 + \frac{2^p}{p^{s-1}}$$

showing that $\|\tilde{\omega}\|_s^s = +\infty$.

EXAMPLE 5.17. Fix a prime p and consider $G = \bigoplus_{n \geq 2} \text{Aff}(p^n)$. Let $\omega \in \text{sp}(Z\ell^1(G))$, and let (χ_n) be the corresponding sequence of group characters. Recall that $\text{Aff}(p^n)$ has a unique non-linear irreducible character, of degree $p^n - 1$, which we shall denote in this example by ψ_n . For each $s \in [1, \infty)$, a short calculation gives

$$\left(\frac{1}{p^n - 1} \|\psi_n\|_s\right)^s = 1 + \frac{1}{(p^n - 1)^{s-1}} \leq 1 + p^{-(s-1)(n-1)}.$$

Now, for each n , either χ_n is linear or $\chi_n = \psi_n$. Let

$$K_\omega := \{n \in \mathbb{N} : \psi_n \chi_n \text{ is linear}\}.$$

If K_ω is infinite, then since $\text{mcr}(\chi_n) = 1$ for all $n \in K_\omega$, it follows from Theorem 5.4 that $\omega \notin c_0$. Suppose, on the other hand, that K_ω is finite, and let $s > 1$. Then by Lemma 5.11,

$$\begin{aligned} \|\omega\|_s^s &= \prod_{n \in \mathbb{N}} d_{\chi_n}^{-s} \|\chi_n\|_s^s = \left(\prod_{n \in K_\omega} p^n (p^n - 1)\right)^s \prod_{n \in \mathbb{N} \setminus K_\omega} \left(\frac{1}{p^n - 1} \|\psi_n\|_s\right)^s \\ &\leq \left(\prod_{n \in K_\omega} p^n (p^n - 1)\right)^s \prod_{n \in \mathbb{N} \setminus K_\omega} \left(1 + p^{-(s-1)(n-1)}\right) \\ &\leq \left(\prod_{n \in K_\omega} p^n (p^n - 1)\right)^s \exp\left(\sum_{n=1}^{\infty} p^{-(s-1)n}\right) \end{aligned}$$

is finite.

Direct calculation shows that when G is as in Example 5.17, none of the algebra characters $\omega \in \text{sp}(Z\ell^1(G))$ lie in ℓ^1 . The following result shows that this is not a failing of our example, but a general result.

THEOREM 5.18. *Let $G = \bigoplus_{i \in \mathbf{I}} G_i$ be an RDPF group, and let $\omega \in \text{sp}(\text{Z}\ell^1(G))$. If G is infinite, then $\omega \notin \ell^1$.*

Proof. Since G is infinite, \mathbf{I} is infinite; we may also suppose without loss of generality that $|G_i| \geq 2$ for all i . Let (χ_i) be the family of group characters corresponding to ω , and for ease of notation let d_i denote the degree χ_i .

Using Lemma 5.11 and the fact that $|G_i| \geq d_i^2$,

$$\|\tilde{\omega}\|_1 = \prod_{i \in \mathbf{I}} d_i^{-1} \|\chi_i\|_1 \geq \prod_{i \in \mathbf{I}} \frac{d_i}{|G_i|} \|\chi_i\|_1. \quad (5.7)$$

Define $F = \{i \in \mathbf{I} : d_i \|\chi_i\|_1 = |G_i|\}$. Since the support of χ_i contains at least one non-identity element, for each $i \in F$ we have $\text{mcr}(\chi_i) = 1$, by Lemma 4.3.

If F is infinite, then $(\text{mcr}(\chi_i))_{i \in \mathbf{I}} \notin c_0(\mathbf{I})$, so by Theorem 5.4 ω does not lie in c_0 , and so cannot lie in ℓ^1 . On the other hand, if F is finite, then $\mathbf{I} \setminus F$ is infinite, and so by combining the inequality (5.7) with Rider's result (Theorem 2.3) we obtain

$$\|\tilde{\omega}\|_1 \geq \prod_{i \in \mathbf{I} \setminus F} \frac{d_i}{|G_i|} \|\chi_i\|_1 \geq \prod_{i \in \mathbf{I} \setminus F} \frac{301}{300} = +\infty,$$

showing that $\omega \notin \ell^1$. □

We finish this section by remarking that most of our examples have been built out of groups of the form $\text{Aff}(q)$. This is to demonstrate that we can achieve a wide range of behaviour (in terms of ℓ^p -norms) while staying within the class of metabelian groups. However, the formulas we have to date suggest that if we merely require algebra characters that lie in ℓ^s where $s \in (2, \infty)$, then we should have many more examples (cf. Example 5.12).

6 ZL-AMENABILITY CONSTANTS FOR GROUPS WITH TWO CHARACTER DEGREES

The last two sections of this paper have a different emphasis, and were originally motivated by attempts to circumvent the use of Rider's theorem (Theorem 2.3) in proving some of our results. These attempts lead naturally to the problem of calculating, or estimating, the amenability constant of $\text{Z}\ell^1(G)$ when G is a finite group, under various hypotheses.

This constant will be denoted, as before, by $\text{AM}(\text{Z}\ell^1(G))$; to reduce needless repetition, it is convenient to call $\text{AM}(\text{Z}\ell^1(G))$ the **ZL-amenability constant** of the finite group G . It has previously been studied by the third author, together with Azimifard and Spronk, in [1], where it is shown ([1, Theorem 1.8]) that

$$\text{AM}(\text{Z}\ell^1(G)) = \frac{1}{|G|^2} \sum_{C, D \in \text{Conj}(G)} \left| \sum_{\chi \in \text{Irr}(G)} \chi(C) \overline{\chi(D)} \right| |C| |D|. \quad (6.1)$$

To explain the notation in this formula: throughout this section, given a finite group G , $\text{Conj}(G)$ denotes the set of conjugacy classes in G , $\text{Irr}(G)$ denotes the set of irreducible

characters on G , and $\chi(C)$ denotes the value taken by a character χ on any (hence every) element of a conjugacy class $C \in \text{Conj}(G)$.

In general, computing the right hand side of (6.1) is difficult: even obtaining useful upper or lower bounds involves some work. However, it simplifies to something more manageable, if all non-linear irreducible representations of G have the same dimension. We shall say that such groups, when non-abelian, have **two character degrees**.

REMARK 6.1. Groups with two character degrees were studied, in an unrelated context, by Isaacs and Passman [3]. (In the terminology of that paper, the groups we consider **have a.c. m** for some integer $m > 1$.) They obtain quite detailed structural information: in particular, G is metabelian [2, Corollary 12.6]. As such, they seem a natural class of examples to consider, if looking for large non-abelian groups whose ZL-amenability constant is reasonably small.

THEOREM 6.2. *Let G be a non-abelian finite group with two character degrees, and let m be the degree of any non-linear irreducible group character of G . Then*

$$\text{AM}(\text{Z}\ell^1(G)) = 1 + 2(m^2 - 1) \left(1 - \frac{1}{|G| |G'|} \sum_{C \in \text{Conj}(G)} |C|^2 \right). \quad (6.2)$$

The advantage of this formula is that, once we know that every non-linear character of G has dimension m , we only need two other pieces of information: the order of the derived subgroup (see Remark 6.3 below) and the size of each conjugacy class. In particular, we do not need the full character table of G .

REMARK 6.3. Given an arbitrary finite group G , let \mathfrak{L} denote the set of linear characters on G . Then \mathfrak{L} is in bijection, in a natural way, with the set of characters on the abelian group $G_{\text{ab}} := G/G'$ (see for instance, [4, Theorem 17.11]; and consequently $|\mathfrak{L}| = |\widehat{G_{\text{ab}}}| = |G_{\text{ab}}| = |G| / |G'|$. We shall use this fact in the proof of Theorem 6.2; it is also more convenient for some of our examples to count the number of linear characters than to work out the derived subgroup.

Before proving Theorem 6.2, we isolate one of the steps as a separate lemma.

LEMMA 6.4. *Let G be an arbitrary finite group, and let $\mathfrak{L} = \{\chi \in \text{Irr}(G) : d_\chi = 1\}$. Then*

$$\frac{1}{|G|^2} \sum_{C, D \in \text{Conj}(G)} |C| |D| \left| \sum_{\chi \in \mathfrak{L}} \chi(C) \overline{\chi(D)} \right| = 1.$$

Proof. Let G_{ab} denote the quotient group G/G' and let $q : G \rightarrow G_{\text{ab}}$ be the quotient homomorphism.

The left-hand side of the putative equation is the norm, in $\text{Z}\ell^1(G \times G)$, of the following idempotent:

$$\widetilde{M} := \frac{1}{|G|^2} \sum_{\chi \in \mathfrak{L}} \chi \otimes \chi.$$

Since each χ is constant on cosets of the derived subgroup, \widetilde{M} factors through the quotient map $Z\ell^1(G \times G) \rightarrow Z\ell^1(G_{\text{ab}} \times G_{\text{ab}})$. For each $x, y \in G$, put $M(q(x), q(y)) := |G'|^2 M(x, y)$; then M is a well-defined element of $Z\ell^1(G_{\text{ab}})$, and a little thought shows that $\|M\| = \|\widetilde{M}\|$. On the other hand, since $\chi \in \mathfrak{L}$ if and only if $\chi = \phi \circ q$ for some $\phi \in \widehat{G_{\text{ab}}}$, (cf. Remark 6.3), we see that

$$M = \frac{1}{|G_{\text{ab}}|^2} \sum_{\phi \in \widehat{G_{\text{ab}}}} \phi \otimes \overline{\phi}$$

Therefore, by [1, Theorem 1.8], M is the unique diagonal element for the Banach algebra $Z\ell^1(G_{\text{ab}}) = \ell^1(G_{\text{ab}})$, so has norm 1. This completes the proof. \square

Proof of Theorem 6.2. To ease notation, we write Conj instead of $\text{Conj}(G)$ throughout this proof. Let

$$\text{AM}_{\text{diag}} = \frac{1}{|G|^2} \sum_{C \in \text{Conj}} |C|^2 \sum_{\chi \in \text{Irr}(G)} d_\chi^2 |\chi(C)|^2$$

and

$$\text{AM}_{\text{off}} = \frac{1}{|G|^2} \sum_{(C,D) \in \text{Conj}^2 : C \neq D} |C| |D| \left| \sum_{\chi \in \text{Irr}(G)} d_\chi^2 \chi(C) \overline{\chi(D)} \right|$$

so that, by (6.1), $\text{AM}(Z\ell^1(G)) = \text{AM}_{\text{diag}} + \text{AM}_{\text{off}}$.

Let $\mathfrak{L} = \{\chi \in \text{Irr}(G) : d_\chi = 1\}$. Recall (see Remark 6.3) that

$$|\mathfrak{L}| = \frac{|G|}{|G'|}; \tag{6.3}$$

moreover, $|G'| \geq \sup_{C \in \text{Conj}} |C|$.

Using Schur column orthogonality, and the fact that $|\chi(\cdot)|^2 = 1$ for every $\chi \in \mathfrak{L}$, we get

$$\begin{aligned} |G|^2 \text{AM}_{\text{diag}} &= \sum_{C \in \text{Conj}} |C|^2 \left(\sum_{\chi \in \text{Irr}(G)} m^2 |\chi(C)|^2 - \sum_{\chi \in \mathfrak{L}} (m^2 - 1) |\chi(C)|^2 \right) \\ &= \sum_{C \in \text{Conj}} |C|^2 \left(m^2 \frac{|G|}{|C|} - (m^2 - 1) |\mathfrak{L}| \right) \\ &= m^2 |G|^2 - (m^2 - 1) |\mathfrak{L}| \sum_{C \in \text{Conj}} |C|^2. \end{aligned} \tag{6.4}$$

Similarly,

$$\begin{aligned}
|G|^2 \text{AM}_{\text{off}} &= \sum_{(C,D): C \neq D} |C| |D| \left| \sum_{\chi \in \text{Irr}(G)} m^2 \chi(C) \overline{\chi(D)} - \sum_{\chi \in \mathfrak{L}} (m^2 - 1) \chi(C) \overline{\chi(D)} \right| \\
&= \sum_{(C,D): C \neq D} |C| |D| \left| \sum_{\chi \in \mathfrak{L}} (m^2 - 1) \chi(C) \overline{\chi(D)} \right| \\
&\quad \text{(by Schur col. orthogonality)} \\
&= (m^2 - 1) \sum_{(C,D)} |C| |D| \left| \sum_{\chi \in \mathfrak{L}} \chi(C) \overline{\chi(D)} \right| - (m^2 - 1) \sum_C |C|^2 |\mathfrak{L}| \\
&\quad \text{(since } |\chi(\cdot)|^2 = 1 \text{ for all } \chi \in \mathfrak{L}) \\
&= (m^2 - 1) |G|^2 - (m^2 - 1) \sum_C |C|^2 |\mathfrak{L}|,
\end{aligned} \tag{6.5}$$

with the last equation following from Lemma 6.4. Combining (6.4) and (6.5), using (6.3), and re-arranging terms, we obtain the desired formula. \square

7 ZL-AMENABILITY CONSTANTS OF PARTICULAR GROUPS

7.1 DIHEDRAL GROUPS

Using Theorem 6.2, we can find the ZL-amenability constants for several well-known families of finite groups. A natural first step is to consider dihedral groups, all of whose non-linear characters have degree 2.

Let us fix some notation: D_n denotes the **dihedral group of order $2n$** , whose standard presentation is

$$D_n = \langle r, s \mid r^n = s^2 = 1, sr = r^{-1}s \rangle.$$

The full character table for D_n is well-known and can be found in standard sources: for instance, see [4, pp. 182–183]. However, as previously remarked, to use our formula we only need the number of linear characters and the cardinalities of the conjugacy classes.

THE CASE OF EVEN n . Suppose $n = 2\nu$ for some integer $\nu \geq 2$. It is known that D_n has four linear characters (so that its derived subgroup has order $n/2$), and all other characters have degree 2. Also, its conjugacy classes has two conjugacy classes of size 1 (namely $\{1\}$ and $\{r^\nu\}$), two of size $n/2$ (namely $[s]$ and $[rs]$), and $\nu - 1$ of size 2 (the remaining rotations, paired up). Thus

$$\sum_{C \in \text{Conj}(D_{2\nu})} |C|^2 = 2 \cdot 1^2 + 2 \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2} - 1\right) \cdot 2^2 = \frac{1}{2}(n^2 + 4n - 4);$$

and so, by our general formula (6.2),

$$\begin{aligned} \text{AM}(D_{2\nu}) &= 1 + 2(2^2 - 1) \left(1 - \frac{n^2 + 4n - 4}{2n^2} \right) = 1 + 6 \frac{n^2 - 4n + 4}{2n^2} \\ &= 1 + 3 \left(1 - \frac{2}{n} \right)^2. \end{aligned} \quad (7.1)$$

THE CASE OF ODD n . Suppose $n = 2\nu + 1$ where ν is an integer ≥ 1 . Then D_n has two linear characters (so that its derived subgroup has order n), and all other characters have degree 2. Also, its conjugacy classes are as follows: the trivial conjugacy class of the identity; the conjugacy class consisting of all involutions, which has size n ; and $(n - 1)/2$ conjugacy classes of size 2 (each consisting of a rotation and its inverse). Thus

$$\sum_{C \in \text{Conj}(D_{2\nu+1})} |C|^2 = 1^2 + n^2 + \frac{n-1}{2} \cdot 2^2 = n^2 + 2n - 1;$$

and so, by our general formula (6.2),

$$\begin{aligned} \text{AM}(D_{2\nu+1}) &= 1 + 2(2^2 - 1) \left(1 - \frac{n^2 + 2n - 1}{2n^2} \right) = 1 + 6 \frac{n^2 - 2n + 1}{2n^2} \\ &= 1 + 3 \left(1 - \frac{1}{n} \right)^2. \end{aligned} \quad (7.2)$$

In fact, when n is odd, D_{2n} fits into a family of more general examples, for which one can simplify (6.2) even further. These groups are the topic of the next subsection.

7.2 FROBENIUS GROUPS WITH ABELIAN COMPLEMENT AND KERNEL

Frobenius groups admit various characterizations or equivalent definitions. The following one is convenient for our purposes.

DEFINITION 7.1 (cf. [12, Theorem 8.2]). A finite group G is a **Frobenius group** if it has a finite, proper, non-trivial subgroup H which is **malnormal**, i.e. which satisfies $H \cap gHg^{-1} = \{e\}$ for all $g \in G \setminus H$. We say that H is a **Frobenius complement** in G .

Given a Frobenius complement $H < G$, define

$$K := \left(G \setminus \bigcup_{g \in G} gHg^{-1} \right) \cup \{e\}.$$

Clearly K is conjugation-invariant: it is a deep result of Frobenius that K is actually a subgroup of G , called the **Frobenius kernel** of G , and that G is the semidirect product $K \rtimes H$. (See Passman's book, in particular the proof of [12, Theorem 17.1], for further details.)

REMARK 7.2. *A priori*, K depends on the particular choice of Frobenius complement H . However, it turns out that if G has a Frobenius complement H and K is the corresponding Frobenius kernel, then K is equal to the Fitting subgroup of G ; moreover, all proper, non-trivial, malnormal subgroups of G are conjugate in G ([12, Corollary 17.5]). These highly non-obvious results are sometimes summarized in the slogan “a finite group can be Frobenius in at most one way”.

For sake of brevity, we shall write “let $G = K \rtimes H$ be Frobenius” as an abbreviation for “let G be a finite Frobenius group, with Frobenius complement H and Frobenius kernel K .”

PROPOSITION 7.3. *Let $G = K \rtimes H$ be Frobenius. Suppose H is an abelian group of order h , and K is an abelian group of order k . Then h divides $k - 1$. Moreover:*

- (i) *G has trivial centre, $(k-1)/h$ conjugacy classes of size h , and $h-1$ conjugacy classes of size k .*
- (ii) *G has exactly h linear characters; the remaining characters each have degree h .*

The proposition is an assembly of several standard facts about Frobenius groups, but it is difficult to locate a reference that states concisely what we need; so a proof is given in Appendix B.

THEOREM 7.4. *Let G be a Frobenius group whose complement and kernel are both abelian; let h and k be the orders of the complement and kernel, respectively. Then*

$$\text{AM}(\text{Z}\ell^1(G)) = 1 + 2 \cdot \frac{h^2 - 1}{h} \left(1 - \frac{h-1}{k}\right) \left(1 - \frac{1}{k}\right) \quad (7.3)$$

Proof. By Proposition 7.3,

$$\sum_{C \in \text{Conj}(G)} |C|^2 = 1 + \frac{k-1}{h} h^2 + (h-1) k^2 = 1 + h(k-1) + (h-1)k^2;$$

and substituting the remaining information from Proposition 7.3 into the general formula (6.2) yields

$$\begin{aligned} \frac{\text{AM}(\text{Z}\ell^1(G)) - 1}{2} &= (h^2 - 1) \left(1 - \frac{1 + h(k-1) + (h-1)k^2}{hk^2}\right) \\ &= \frac{h^2 - 1}{h} \cdot \frac{-1 - hk + h + k^2}{k^2} \\ &= \frac{h^2 - 1}{h} \left(1 - \frac{h}{k} + \frac{h-1}{k^2}\right); \end{aligned}$$

factorizing and rearranging this gives the formula (7.3), as required. \square

EXAMPLE 7.5 (Dihedral groups of odd order, revisited). Let n be an odd integer with $n \geq 3$. Using the standard presentation of D_n as given earlier, we see that the subgroup

generated by the ‘reflection’ t is malnormal, while the Frobenius kernel turns out to be the subgroup generated by the ‘rotation’ r . Putting $h = 2$ and $k = n$ in (7.3) gives

$$\text{AM}(\text{Z}\ell^1(D_n)) = 1 + 3 \left(1 - \frac{1}{n}\right)^2,$$

just as we had before.

EXAMPLE 7.6 (Affine groups of finite fields). Let \mathbb{F}_q be a finite field of order q , where q is a prime power ≥ 3 . The affine group of \mathbb{F}_q has already been introduced (see Example 5.7). It is straightforward to check that the subgroup of $\text{Aff}(\mathbb{F}_q)$ corresponding to the multiplicative group of \mathbb{F}_q is a proper, non-trivial, malnormal subgroup; the Frobenius kernel turns out to be the normal subgroup of $\text{Aff}(\mathbb{F}_q)$ corresponding to the additive group of \mathbb{F}_q . Both are abelian, so we can apply Theorem 7.4: substituting the appropriate values into (7.3), we get

$$\begin{aligned} \text{AM}(\text{Z}\ell^1(\text{Aff}(\mathbb{F}_q))) &= 1 + 2 \cdot \frac{(q-1)^2 - 1}{q-1} \left(1 - \frac{q-2}{q}\right) \left(1 - \frac{1}{q}\right) \\ &= 1 + 2 \cdot \frac{2q - q^2}{q-1} \cdot \frac{2}{q} \cdot \frac{q-1}{q} \\ &= 1 + 4 \cdot \frac{q-2}{q} \\ &= 5 - \frac{8}{q}. \end{aligned}$$

REMARK 7.7. For all odd primes p , $2 \leq \text{AM}(\text{Aff}(p)) \leq 5$, while $\text{Aff}(p)$ has an irreducible representation of dimension $p-1$. This shows that the amenability constant of $\text{Z}\ell^1(G)$ cannot be bounded from below by an increasing function of $\max(d_\chi : \chi \in \text{Irr}(G))$.

EXAMPLE 7.8 ($a^2x + b$ groups). Let q be an odd prime power ≥ 5 , and let $d = (q-1)/2$. Consider the following subgroup of $\text{Aff}(\mathbb{F}_q)$, sometimes referred to as the “ $a^2x + b$ group over \mathbb{F}_q ”:

$$G_q := \left\{ \begin{pmatrix} a^2 & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\}.$$

Recalling that \mathbb{F}_q^\times is cyclic, pick a generator z , and let H be the subgroup of $\text{Aff}(\mathbb{F}_q)$ generated by $\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$. H turns out to be malnormal, and so G_q is Frobenius; the Frobenius kernel K turns out to be the normal subgroup corresponding to the additive group of \mathbb{F}_q . So both K and H are abelian; the former has order q while the latter has order d , so using (7.3) we get

$$\begin{aligned} \text{AM}(\text{Z}\ell^1(G_q)) &= 1 + 2 \cdot \frac{d^2 - 1}{d} \left(1 - \frac{d-1}{q}\right) \left(1 - \frac{1}{q}\right) \\ &= 1 + 2 \cdot \frac{q^2 - 2q - 3}{2(q-1)} \cdot \frac{q+3}{2q} \cdot \frac{q-1}{q} \end{aligned}$$

which simplifies to

$$\text{AM}(\text{Z}\ell^1(G_q)) = 1 + \frac{q+1}{2} \left(1 - \frac{9}{q^2}\right) \quad (7.4)$$

As a consistency check: when $q = 5$, Equation (7.4) gives $\text{AM}(\text{Z}\ell^1(G_5)) = 73/25$. On the other hand, it is straightforward to check that G_5 is isomorphic to the dihedral group of order 10, and using our earlier formulas we have $\text{AM}(\text{Z}\ell^1(D_5)) = 73/25$.

REMARK 7.9. Even though G_q is a (index 2) subgroup of $\text{Aff}(\mathbb{F}_q)$, it may have a larger ZL-amenability constant. Indeed, it is clear from the formulas obtained in Examples 7.6 and 7.8 that

$$\lim_{q \rightarrow \infty} \text{AM} \text{Z}\ell^1(\text{Aff}_q) = 5 \quad \text{while} \quad \lim_{q \rightarrow \infty} q^{-1} \text{AM} \text{Z}\ell^1(G_q) = \frac{1}{2}$$

Example 7.8 shows that within the class of groups with two character degrees, we can obtain arbitrarily large ZL-amenability constants. It is natural to ask how small such constants can be; however, it seems difficult to obtain universal lower bounds using the formula (6.2) for the ZL-amenability constant. (We already know that the ZL-amenability constant of a nonabelian finite group is bounded below by $301/300$, but this requires the use of Rider's theorem on norms of idempotents (Theorem 2.3) and seems unlikely to be sharp.)

For Frobenius groups with abelian complement and kernel, we can be much more precise.

THEOREM 7.10. *Let G be a Frobenius group with abelian complement and kernel. Then $\text{AM}(\text{Z}\ell^1(G)) \geq 7/3$, with equality if and only if G is the dihedral group of order 6.*

Proof. Let h be the order of the Frobenius complement of G , and k the order of its Frobenius kernel. Note that G is isomorphic to D_3 if and only if $h = 2$ and $k = 3$.

To reduce notational clutter, let $F(k, h)$ denote $\frac{1}{2}(\text{AM}(\text{Z}\ell^1(G)) - 1)$. Then by Theorem 7.4,

$$F(k, h) = \frac{h^2 - 1}{h} \left(1 - \frac{h - 1}{k}\right) \left(1 - \frac{1}{k}\right) \quad (\clubsuit)$$

and it suffices to prove that $F(k, h) \geq 2/3$ with equality if and only if $(h, k) = (2, 3)$ (subject to h and k arising from a Frobenius group of the specified form).

Note that for fixed h , $F(\cdot, h)$ is a strictly increasing function. As observed above, h divides $k - 1$, so in particular $k \geq h + 1$; hence, $F(k, h) \geq F(h + 1, h)$, with equality if and only if $h = k - 1$. Direct calculation gives

$$F(h + 1, h) = \frac{h^2 - 1}{h} \cdot \frac{2}{h + 1} \cdot \frac{h}{h + 1} = \frac{2(h - 1)}{h + 1} = 2 \left(1 - \frac{2}{h + 1}\right),$$

and so $F(h + 1, h) \geq 2/3$, with equality if and only if $h = 2$. This completes the proof. \square

If we consider more general groups with two character degrees, then there is an infinite family of such groups whose ZL-amenability constants are less than 2. This will be seen in the next and final subsection of the paper.

7.3 EXTRA-SPECIAL p -GROUPS.

These were described earlier in Example 4.5. Let G be an extraspecial group of order p^{2n+1} , where p is a prime. Then all non-linear, irreducible group characters of G have degree p^n ; we also saw that $G' = Z(G)$ has order p . To apply Theorem 6.2, we need also to know the sizes of the conjugacy classes: it turns out that, apart from the elements of the center, the conjugacy classes in the group are the non-trivial cosets of the derived subgroup, so there are $p^{2n} - 1$ of these, each with size p .

Therefore

$$\sum_{C \in \text{Conj}(G)} |C|^2 = p \cdot 1^2 + (p^{2n} - 1) \cdot p^2 = p^{2n+2} - p^2 + p$$

and so Theorem 6.2 gives

$$\begin{aligned} \text{AM}(Z\ell^1(G)) &= 1 + 2(p^{2n} - 1) \left(1 - \frac{p^{2n+2} - p^2 + p}{p^{2n+2}} \right) \\ &= 1 + 2 \left(1 - \frac{1}{p^{2n}} \right) \left(1 - \frac{1}{p} \right) \end{aligned} \tag{7.5}$$

In particular, if G is an extraspecial group of order 2^{2n+1} , then $\text{AM}(Z\ell^1(G)) = 2 - 2^{-2n}$, giving an infinite family of finite groups G for which $1 < \text{AM}(Z\ell^1(G)) < 2$.

REMARK 7.11. Within the class of extra-special p -groups, the ZL-amenability constant is minimized when we take $p = 2$ and $n = 1$. This example is nothing but the dihedral group of order 8, whose amenability constant is $7/4$.

This is the smallest ZL-amenability constant we have found for non-abelian groups; in contrast, the best known lower bound for

$$\inf\{\text{AM}(Z\ell^1(G)) : G \text{ is finite and non-abelian}\}$$

is $1+1/300$, as obtained via Rider's result. See [1, corollary 1.9]. It would be of interest to obtain a stronger lower bound, or to find a non-abelian finite group G for which $\text{AM}(Z\ell^1(G)) < 7/4$.

SUMMARY INFORMATION FOR SECTION 7

Explanation of table:

- “Ref.” gives the number of the relevant theorem, example or equation.
- \mathfrak{L} is the set of linear characters.
- “c.d.” stands for the character degree of the non-linear characters.
- “min.” denotes the minimum value of $\text{AM}(Z\ell^1(G)) - 1$ within the specified family of groups.

Ref.	G	$ G $	$ \mathcal{L} $	c.d.	$\text{AM}(\mathbb{Z}\ell^1(G)) - 1$	min.
Ex. 7.6	$\text{Aff}(\mathbb{F}_q),$ $q \geq 3$	$q(q-1)$	$q-1$	$q-1$	$4(1-2q^{-1})$	$4/3$
Ex. 7.8	$ax^2 + b$ of $\mathbb{F}_q,$ $q \geq 5$	$q(q-1)/2$	$\frac{1}{2}(q-1)$	$q-1$	$\frac{1}{2}(q+1)(1-9q^{-2})$	$48/25$
Ex. 7.5	$D_n,$ n odd ≥ 3	$2n$	2	2	$3(1-n^{-1})^2$	$4/3$
Eq. (7.1)	$D_n,$ n even ≥ 4	$2n$	4	2	$3(1-(2n)^{-1})^2$	$3/4$
Eq. (7.5)	extraspecial	p^{2n+1}	p^{2n}	p^n	$3(1-p^{-2n})(1-p^{-1})$	$3/4$

Figure 3: Summary table for some groups with two character degrees

A APPENDIX: A PROOF THAT 1-MINIMAL GROUPS ARE NILPOTENT

In this appendix we present a proof of Theorem 4.6, namely that a 1-minimal finite group must be nilpotent. The argument is paraphrased from one shown to the second author by F. Ladisch [7], and is included here with his kind permission. We have tried to make the presentation accessible to non-specialists in character theory.

Let us recall some of the relevant definitions and basic facts. When χ is an irreducible group character on a finite group G , the centre of χ , denoted by $\mathbf{Z}(\chi)$ of χ is a normal subgroup of G . We say χ is 1-minimal if and only if $\mathbf{Z}(\chi) = \text{supp}(\chi)$.

The following lemma, which appears to be well known to specialists, uses no special properties of G .

LEMMA A.1. *Let N be a normal subgroup of G and let χ be an irreducible character of G . If N contains a non-identity element, then so does $N \cap \text{supp}(\chi)$.*

Proof. Consider the character $\chi|_N$. We have two cases to consider. If $\chi|_N$ is not orthogonal to the trivial character ε_N , then since N is normal it follows from a theorem of Clifford that χ is proportional to ε_N (see, e.g. [2, Corollary 6.7]). In particular, $N \cap \text{supp}(\chi) = N$ contains a non-identity element.

On the other hand, if $\chi|_N$ is orthogonal to ε_N , then $0 = \langle \varepsilon_N, \varepsilon_N \rangle = \frac{1}{|N|} \sum_{x \in N} \chi(x)$. Since $\chi(e) > 0$, we conclude that χ is non-zero on at least one non-identity element of N . \square

COROLLARY A.2. *Let G be a group with at least two elements, and let S be a set of 1-minimal characters on G . Then $\bigcap_{\chi \in S} \mathbf{Z}(\chi)$ contains a non-identity element.*

Proof. We induct on the size of S . If S is empty there is nothing to prove. Otherwise, suppose $\chi_1, \dots, \chi_{n-1}$ are 1-minimal characters for which $N := \mathbf{Z}(\chi_1) \cap \dots \cap \mathbf{Z}(\chi_{n-1})$ contains a non-identity element. N is a normal subgroup of G , since $\mathbf{Z}(\chi_i)$ is for each i ; and since χ_n is 1-minimal, $\text{supp}(\chi) = \mathbf{Z}(\chi_n)$. By Lemma A.1, $N \cap \mathbf{Z}(\chi_n)$ therefore contains a non-identity element, completing the inductive step. \square

Proof of Theorem 4.6. We argue by strong induction on the order of the group. Every group of order ≤ 5 is abelian, hence in particular both 1-minimal and nilpotent.

Let $n \geq 6$ and suppose inductively that all 1-minimal groups of order $< n$ are nilpotent. Let G be a 1-minimal group of order n . Now $Z(G) = \bigcap_{\chi \in \text{Irr}(G)} \mathbf{Z}(\chi)$ – this is true for *any* finite group, see [2, Corollary 2.28] – and therefore by Corollary A.2, $Z(G)$ contains a non-identity element. Since quotients of 1-minimal finite groups are themselves 1-minimal, $G/Z(G)$ is 1-minimal and has order $\leq n/2 < n$, and so it is nilpotent by the inductive hypothesis. But then G is a central extension of a nilpotent group, hence is itself nilpotent, as required. \square

B PROPERTIES OF FROBENIUS GROUPS

In this appendix we collect the facts about Frobenius groups that are needed to prove Proposition 7.3. Since we are interested only in the special case where both complement and kernel are abelian, it will sometimes be easier to give short proofs, than to cite general results and then specialize. On the other hand, in some places we shall merely give appropriate references to the literature. Our notation differs slightly from that of Sections 6 and 7, in that the conjugacy class of x in G will be denoted by x^G , and the centralizer of x in G will be denoted by $C_G(x)$.

Throughout, $G = K \rtimes H$ is Frobenius; k and h denote the orders of K and H respectively. By considering the permutation action of G on cosets of H , it follows easily from the malnormal property that h divides $k - 1$. (This does not need Frobenius's result that K is a group.)

Proof of Proposition 7.3 (i). We repeatedly use the fact that each element of G can be written either as xb where $x \in K$ and $b \in H$, or as ay where $a \in H$ and $y \in K$. (This is immediate from the decomposition of G as a semidirect product of H and K .)

The first step is to identify the conjugacy classes of K inside G . The malnormal property implies that, for each $x \in K \setminus \{e\}$, $C_G(x) \cap H = \{e\}$. If K is abelian, it follows (since $G = HK$) that $C_G(x) = K$, and therefore that $|x^G| = |G|/|K| = h$, for all $x \in K \setminus \{e\}$.

For the second step, recall that by definition, $G \setminus K = \left(\bigcup_{g \in G} gHg^{-1} \right) \setminus \{e\}$. If H is abelian, then (since $G = KH$) we obtain

$$G \setminus K = \left(\bigcup_{x \in K} xHx^{-1} \right) \setminus \{e\}.$$

Now, by malnormality of H inside G , and the fact that $H \cap K = \{e\}$, we see that: whenever $x, y \in K$ and $x \neq y$, then $xHx^{-1} \cap yHy^{-1} = \{e\}$. Thus, the function $H \setminus \{e\} \rightarrow \text{Conj}(G)$, $a \mapsto a^G$, is injective, and $|a^G| = k$ for each $a \in H \setminus \{e\}$. This gives us the required partition of $G \setminus K$ into $h - 1$ disjoint conjugacy classes, each of size k . \square

To prove the second part of Proposition 7.3, we appeal to some general results on the character theory of Frobenius groups.

PROPOSITION. *Let $G = K \rtimes H$ be Frobenius. The following are irreducible characters on G :*

- *the characters arising by composing irreducible characters of H with the quotient map $G \rightarrow G/K \cong H$;*
- *the characters arising by inducing an irreducible character of K up to G .*

Moreover, every irreducible character of G arises in this way.

Proof. See, for example, [2, Theorem 6.34]. □

In the special case where H and K are abelian, it follows immediately that G has two character degrees. The irreducible characters of G that arise by inducing irreducible characters from K all have degree $|G : K| = h$; the remaining characters are all linear, arising from the irreducible characters of H , and there are precisely $|\widehat{H}| = |H| = h$ of them. This completes the proof of Proposition 7.3 (ii).

ACKNOWLEDGEMENTS

Some of the results here are taken from the first author's PhD thesis, which is supported by a Dean's Scholarship from the University of Saskatchewan. The second author was partially supported by NSERC Discovery Grant 366066-2009 and the third author by NSERC Discovery Grant 402153-2011.

The second author thanks F. Ladisch for useful exchanges, in particular for pointing out that extraspecial p -groups provided examples of non-abelian 1-minimal groups.

REFERENCES

- [1] A. AZIMIFARD, E. SAMEI, AND N. SPRONK, *Amenability properties of the centres of group algebras*, J. Funct. Anal., 256 (2009), pp. 1544–1564.
- [2] I. M. ISAACS, *Character theory of finite groups*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Pure and Applied Mathematics, No. 69.
- [3] I. M. ISAACS AND D. S. PASSMAN, *A characterization of groups in terms of the degrees of their characters. II*, Pacific J. Math., 24 (1968), pp. 467–510.
- [4] G. JAMES AND M. LIEBECK, *Representations and characters of groups*, Cambridge University Press, New York, second ed., 2001.
- [5] B. E. JOHNSON, *Non-amenability of the Fourier algebra of a compact group*, J. London Math. Soc. (2), 50 (1994), pp. 361–374.
- [6] E. KANIUTH, A. T. LAU, AND J. PYM, *On character amenability of Banach algebras*, J. Math. Anal. Appl., 344 (2008), pp. 942–955.

- [7] F. LADISCH, *Characters on a finite group with ‘extremal’ behaviour*. MathOverflow. <http://mathoverflow.net/questions/56539> (version: 2011-02-24).
- [8] F. I. MAUTNER, *The regular representation of a restricted direct product of finite groups*, Trans. Amer. Math. Soc., 70 (1951), pp. 531–548.
- [9] M. S. MONFARED, *Character amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc., 144 (2008), pp. 697–706.
- [10] R. D. MOSAK, *The L^1 - and C^* -algebras of $[FIA]_B^-$ groups, and their representations*, Trans. Amer. Math. Soc., 163 (1972), pp. 277–310.
- [11] B. H. NEUMANN, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc. (3), 1 (1951), pp. 178–187.
- [12] D. PASSMAN, *Permutation groups*, W. A. Benjamin, Inc., New York – Amsterdam, 1968.
- [13] D. RIDER, *Central idempotent measures on compact groups*, Trans. Amer. Math. Soc., 186 (1973), pp. 459–479 (1974).
- [14] U. STEGMEIR, *Centers of group algebras*, Math. Ann., 243 (1979), pp. 11–16.
- [15] A. TERRAS, *Fourier analysis on finite groups and applications*, vol. 43 of London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 1999.

Mahmood Alaghmandan, Yemon Choi, and Ebrahim Samei

Department of Mathematics and Statistics

McLean Hall

University of Saskatchewan

Saskatoon, SK

Canada S7N 5E6

E-mails: mahmood.a@usask.ca, choi@math.usask.ca, samei@math.usask.ca